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Optimal Macroprudential Policy and Asset Price Bubbles

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Optimal Macroprudential Policy and Asset Price Bubbles*

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Abstract

We study the interplay between firms' indebtedness and stock market bubbles. A bubble relaxes borrowing constraints and increases borrowing capacity of credit-constrained firms. Yet a deflating bubble amplifies downturns when constraints start binding. We show analytically and quantitatively that optimal macroprudential policy should respond to bubbles in a non-monotonic way, which depends on the underlying level of indebtedness. If the level of debt is moderate, policy should accommodate the bubble to reduce the incidence of a binding borrowing constraint. If debt is elevated, policy should lean against the bubble more aggressively to mitigate the externalities when constraints bind.

JEL Classification: E2, E44, G1

Keywords: borrowing constraints, rational bubbles, nonfinancial business borrowing, macroprudential regulation, optimal policy

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1 Introduction

In the aftermath of the global financial crisis (GFC), policymakers and academics have widely shared the view that policy should lean against rapid credit growth. As a result, new macroprudential instruments to tackle credit imbalances have been introduced in many countries, including counter-cyclical capital buffers for banks and loan-to-value ratios for housing loans, and a vast literature studies their efficacy.

Many papers have shown that surges in credit are often accompanied by hikes in asset prices, and that the interplay between the two may have large economic effects. For example, Mishkin (2011) and Jordà, Schularick and Taylor (2015) argue that credit-fueled asset price bubbles are more dangerous to financial stability and economic growth than bubbles not followed by debt build-ups. Fostel and Geanakoplos (2008) and Adrian and Shin (2009) show that the feedback loop between asset prices and credit can lead to procyclical leverage and to financial instability, while Greenwood, Hanson, Shleifer and Sorensen (2020) show that the combination of rapid credit and asset price growth, whether in the nonfinancial business or in the household sector, increases substantially the probability of a financial crisis. On the contrary, in an influential paper, Bernanke and Gertler (1999) argue that central banks should not lean against asset price bubbles, though their focus was on monetary rather than macroprudential policy. Overall, there is no broadly accepted view on whether macroprudential policy should respond to asset prices *beyond* its response to tackle credit imbalances (Barlevy, 2018). One reason for this may have been the modeling difficulties with incorporating asset price bubbles into theoretical models suitable for performing optimal policy analysis.[1]

We contribute to this discussion by proposing a theoretical framework that integrates a meaningful policy analysis into a model of asset price overvaluations.[2] We use this framework to study the following questions: Should macroprudential policy respond to asset price overvaluations over and beyond its role in tackling excessive levels of credit? If yes, should macroprudential policy be more aggressive or more accommodative? Finally, to what extent does the optimal policy response to asset price overvaluations depend on the level of debt in the economy?

We focus on overvaluations in the stock market and study the interplay between equity prices and business credit, which has received less attention than the feedback loop between credit and overvaluations in real estate.[3] This lack of attention may be due to the fact that the GFC centered around a surge in housing prices and mortgage credit, while it has been commonly argued that the dot-com bubble of the early 2000s did not have severe real consequences as it was not credit-fueled (Dell'Ariccia et al. 2011). However, today's environment is markedly different: Historically-high corporate indebtedness and elevated valuations in the stock market are seen

[1] Additionally, concerns about timely and precise identification of overvaluations have made many policymakers reluctant to react to rapid asset price growth. This issue is, arguably, relatively less concerning in the aftermath of the GFC, as policymakers and academics have placed more effort on detecting "valuation pressures" in asset prices. Some examples include the Office of Financial Research's Financial System Vulnerabilities Monitor and the Shiller CAPE index (Cyclically Adjusted Price Earnings).

[2] We define asset price overvaluation as a positive deviation of the market price from its fundamental value. In the rest of the paper, we will use the terms asset price bubbles and price overvaluations interchangeably.

[3] The mechanism is also relevant for the interaction of real estate overvaluations and credit imbalances, but the model would need to be augmented to show this channel formally.

as two key financial vulnerabilities. Figure 1 illustrates this for the United States. The chart on the left plots the mortgage credit-to-GDP gap (a measure of imbalances for mortgage credit) along with the S&P/Case-Shiller U.S. National Home Price Index. Both credit imbalances and housing overvaluations were increasing sharply in the build-up of the GFC, but while house prices have continued to increase in recent years, housing credit remains subdued. The situation is different for corporate borrowing and stock market overvaluations, as shown in the chart on the right that plots the credit-to-GDP gap for nonfinancial corporate debt along with the Shiller Cyclically-Adjusted Price Earnings (CAPE) index. While corporate credit growth was only somewhat above the trend in the run-up to the GFC (and the dot-com bubble before), it is clearly elevated now, and it is accompanied by elevated stock market valuations.

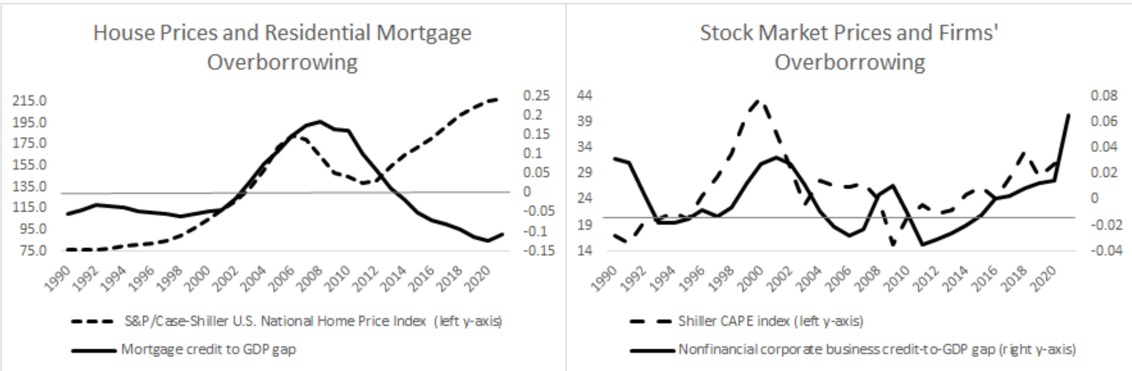

Figure (1) U.S. Credit Imbalances and Asset Price Overvaluations.

Note: Data on mortgage credit and nonfinancial corporate business credit are taken from the Financial Accounts of the U.S. GDP and the S&P/Case-Shiller U.S. National Home Price Index are taken from the FRED database, while data for the Shiller CAPE index are taken from Robert Shiller's website. The credit-to-GDP gap is computed based on a smooth trend obtained through the Hodrick-Prescott filter with a smoothing parameter $\lambda = 400,000$ (after Basel Committee's guidelines for setting the counter-cyclical capital buffer). Large positive (negative) values indicate excessive (subdued) credit.

Our paper illustrates how the feedback between the two generates a market failure requiring a policy intervention. Moreover, we highlight how the policy response should differ when both borrowing and valuations are elevated, as in the current environment, compared to when overvaluations are not accompanied by large credit imbalances, as during the dot-com bubble.

To study the interplay between firms' overborrowing and stock market overvaluations, we develop a dynamic stochastic general equilibrium model with an occasionally binding borrowing constraint and a rational stock price bubble, building on the work of Bianchi and Mendoza (2018) (henceforth Bianchi-Mendoza) and Miao and Wang (2018) (henceforth, Miao-Wang). We opt to build on these two papers because the former establishes a clear role for macroprudential policy, while the later formally derives a rational stock price bubble.[4] We then solve for the optimal time-consistent macroprudential policy of a planner who cannot commit to future policies, and derive analytically the policy instrument in the form of a tax on (new) borrowing that decentralizes the planner's allocations. Finally, we solve the model numerically by employing global solution methods to show how macroprudential policy should account for asset price overvaluations over the credit cycle.

[4]While these two strands of the literature have developed in parallel, there is a great added value in putting them together to understand when and how macroprudential policy should tackle asset price bubbles, particularly in the current context of elevated corporate debt and stretched stock market valuations.

The economy consists of firms owned by households. Firms borrow from external financiers, but due to lack of commitment, their borrowing capacity is constrained. Unlike Kiyotaki and Moore (1997) and Jermann and Quadrini (2012), who assume that borrowing is limited by the liquidation value of physical capital, we consider a setup, similar to Miao-Wang, where the total value of the firm can be pledged as collateral. We show that the latter borrowing constraint can arise endogenously from an incentive compatibility constraint in an optimal contracting problem between borrowers and lenders. The underlying idea is that lenders cannot only confiscate the physical collateral but also seize the ownership rights over the firm's operations if the firm does not honor its debt obligations (the lenders can then hire a new manager and sell the restructured firm in the equity market). Hence, to the lenders the collateral value is equal to (a portion of) the market, going-concern, value of the firm, which may not only include all the discounted future cashflows but also a bubble component. This type of borrowing constraint is supported by recent empirical evidence on corporate borrowing. Lian and Ma (2020) show that 80 percent of corporate debt in the U.S. is not tied to specific assets, but rather to continuing operations of a restructured firm, while Kermani and Ma (2020) show that liquidation values of listed firms' tangible assets account for a small portion of firms' debt outstanding, and debt enforcement goes beyond focusing on the liquidation value of discrete assets.

Before turning to our results, we discuss our two methodological contributions. The first relates to the conditions required for a bubble's existence. As shown in Miao-Wang, borrowing constraints based on the going-concern value of the firm generate a liquidity premium when they bind, which is key for the existence of the bubble (without violating the transversality condition). While in Miao-Wang the liquidity premium is strictly positive in every period, in our model this is not the case as it is strictly positive only occasionally. To show that the bubble can still exist even under an occasionally positive liquidity premium, we study the asymptotic behavior of the economy and show that the borrowing constraint binds infinitely often despite the presence of a bubble; in turn, this is sufficient for the bubble to exist without violating the transversality condition. In other words, a rational bubble can exist if it continues to be useful to relax borrowing constraints in the future even if the constraint does not always bind.

Our second methodological contribution is to construct a solution algorithm for the global dynamics of indebtedness and bubbly valuations, which is essential to properly model the non-linearities and non-monotonicities introduced by the occasionally binding borrowing constraints. We consider a non-stationary stochastic bubble that emerges exogenously at some point in time, but thereafter grows endogenously and may burst in every period with positive probability (Blanchard and Watson, 1982; Weil, 1987). These features make the bubble an atypical state variable because its value is not predetermined, but rather jointly determined with (current period) consumption as the bubble is priced with agents' stochastic discount factor. This complicates the recursive representation of the equilibrium and the characterization of the global dynamics. We address this issue by introducing an auxiliary step that disciplines the expectations about the future bubble state into the otherwise mainstream global, non-linear, solution algorithm of Bianchi-Mendoza.[5]

[5]Martin and Ventura (2012) also solve for recursive equilibria with rational bubbles, but in their case the value of the bubble in every period is drawn by an exogenous process. As a result, bubbles in their model are equivalent to exogenous states, which simplifies the characterization of a recursive equilibrium.

We now turn to our results. Occasionally binding constraints that depend on endogenous asset values generate pecuniary externalities and justify macroprudential policy interventions.[6] We show that the going concern value of the firm is equal to the (endogenous) value of existing physical capital, which incorporates the discounted value of all future cash-flows, as well as the (endogenous) collateralizeable value of the bubble. Private agents do not internalize how their borrowing decisions affect the price of capital, giving rise to an adverse feedback loop between its collateral value and deleveraging when borrowing constraints bind. The bubble directly affects the incidence of a binding borrowing constraint and of this fundamental pecuniary externality in two ways. On the one hand, the bubble provides additional collateral and can make, otherwise binding, borrowing constraints slack. On the other hand, the bubble may burst and increase the chances that the constraint binds if it has allowed agents to borrow more in the past. We call these combined channels the *extensive margin* through which bubbly boom and bust dynamics affect real outcomes. The extensive margin operates via the occasionally binding nature of the constraint and should extend beyond the specific modeling of bubbles in our paper (see Martin and Ventura, 2018, for a survey of the macroeconomic of bubbles and alternative modeling approaches that the extensive margin may extend to).

Nevertheless, the endogenous bubble valuation in the model introduces novel additional channels through which the bubble affects optimal policy. First, the endogenously determined rate of growth of the bubble depends on production and consumption choices. This introduces an additional externality as private agents do not internalize how their choices affect the future value of the bubble, which is a state variable and, thus, matters for current outcomes through its effect on expectations about future outcomes. Second, only a portion of the bubble value can be pledged as collateral. The collateral value of the bubble is endogenous and is inversely related to the tightness of the borrowing constraint. While private agents internalize how the utilization of factors of production matters for the tightness of the constraint, they fail to understand how the latter matters for the portion of the bubble that can be pledged as collateral. We group these two additional channels, which accrue from the endogenous bubble growth and pledgeability, into what we call the *intensive margin*.

A social planner internalizes the effects of borrowing decisions on the incidence and tightness of binding borrowing constraints, and can choose a different level of borrowing to address the aforementioned externalities. The planner faces the following trade-off: Higher borrowing pushes *current physical-capital and bubble valuations* up, alleviating the negative effects of a binding borrowing constraint today, but also dampens *future valuations*, thereby exacerbating the negative effects of binding borrowing constraints in the future. The relative strength between the two opposing effects determines the level of borrowing implemented by the planner.

The allocations chosen by the planner can be decentralized by a tax or a subsidy on borrowing. Importantly, if the borrowing constraint does not bind today, then the only objective is to alleviate externalities from potentially binding constraints in the future, which calls for a positive tax on borrowing today. In this case, the tax is interpreted as a purely macroprudential tax since it is imposed to address overborrowing during good times (i.e. when the borrowing

[6] This has been emphasized by Bianchi-Mendoza among others. Also, Guerrieri and Iacoviello (2017) study the role of occasionally binding borrowing constraints in generating asymmetric effects of house prices on economic activity, but do not study optimal policy.

constraint does not bind) in order to alleviate the costs from deleveraging during bad times (i.e. when the borrowing constraint becomes binding in the future).

Our key result suggests that the overall effect of the bubble on the optimal macroprudential policy is non-monotonic and depends on the underlying economic conditions. This is due to the fact that the extensive and intensive margins can operate in opposite directions, hence it is not clear whether, on net, the macroprudential tax should be higher or lower in the presence of the bubble. In particular, the extensive margin pushes the tax in the opposite direction and dominates the intensive margin when today's level of debt is moderate. This means that, while the bubble relaxes the borrowing constraint, the externalities associated with the intensive margin are not so severe as agents will not need to deleverage much, if the constraint binds in the future. Hence, macroprudential policy should be more accommodative, and the macroprudential tax should be lower when the bubble is present for a given level of credit imbalances (proxied by the current level of debt). However, as credit imbalances grow, the negative externalities (intensive margin) start to dominate and the macroprudential tax increases to a much higher levels than in the bubbleless economy. Overall, asset overvaluations amplify externalities from high levels of credit imbalances, but at the same time they mitigate the adverse effects when imbalances are at a moderate level.

The above results have important implications for the determination of countercyclical policies targeting credit imbalances employed by regulators globally. In particular, asset overvaluations should not only be used as an argument to lean more aggressively against the wind, but could also imply that regulators need not to worry as much about the build-up of price overvaluations, particularly if the extensive margin dominates.

In terms of our quantitative results, we calibrate our model using OECD data and set the parameters governing the dynamics of the bubble to match the stylized facts in Jordà, Schularick and Taylor (2015). We find that the macroprudential tax in the presence of a bubble increases steadily as credit imbalances grow, and can be as high as three times the tax in the bubbleless economy—which is about 3 percent for the constellation of exogenous states we report result for. Moreover, the bubble allows current credit imbalances, as measured by the debt-to-GDP ratio, to increase more compared to the bubbleless case, before the current borrowing constraint starts binding.

When we simulate the economy, we find that, for a low initial level of debt and an adverse productivity shock, for which the borrowing constraint binds, the reduction in consumption is smaller in the presence of a bubble. However, if the initial level of debt is high, the reduction in consumption is higher with the bubble, which is in line with the result that the intensive margin is stronger when the outstanding debt level is high.[7] Starting from a level of debt equal to the long-run average of the bubbleless economy, we find that the net welfare effect of introducing a bubble is negative. That is, the negative effect from the bubble-induced externalities outweighs the positive effect from relaxing the borrowing constraint. The optimal tax on borrowing, averaging close to 2 percent, increases welfare by 0.4 percent, as measured by compensating consumption variations. Given that optimal taxes may seem hard to implement in practice,

[7] As we discuss later, the presence of a stochastic bubble that cannot re-emerge does not allow us to compute an ergodic distribution that is independent of the initial debt level. However, we find the simulation exercises useful, as they highlight the non-monotonic effects of the bubble.

we also examine the welfare gains accruing from simple tax rules. We find that simple rules can yield larger welfare gains when they account for both credit imbalances and asset price overvaluations.

The rest of this paper is organized as follows. Section 2 reviews the related literature. Section 3 presents the baseline model. We analyze optimal policy in section 4. We show the numerical results in section 5. Finally, section 6 concludes.

2 Literature

Our paper is related to two main strands in the literature. First, it contributes to the literature on rational asset price bubbles. Second, it contributes to the literature on optimal macroprudential policy. While these two areas of research have evolved in parallel, important connections remain to be addressed and a rigorous normative analysis has been lacking. Hence, the main focus of our paper is the design of optimal macroprudential policies in the presence of asset price bubbles and occasionally binding borrowing constraints.

The papers in the literature on rational asset price bubbles differ in the friction that allows a bubble to exist in equilibrium, which matters also for how the bubble affects economic outcomes (Barlevy, 2018). In the early literature, including the seminal works by Samuelson (1958) and Tirole (1985), the bubble exists in equilibrium because of dynamic inefficiency. However, as argued in Abel, Mankiw, Summers and Zeckhauser (1989), real economies are dynamically efficient.[8] Hence, most of the recent papers on rational bubbles have turned away from dynamic inefficiency and consider other frictions to motivate bubbles' existence.

Some models attribute the bubbles' presence to financial frictions. In this class of models, an intrinsically worthless asset or a bubbly component of a productive asset can relax financial frictions by allowing agents to borrow more. Early examples include Kocherlakota (1992) and Santos and Woodford (1997). More recent work has emphasized the role of entrepreneurs and firms facing borrowing constraints, including Kocherlakota (2009), Farhi and Tirole (2012*b*), Martin and Ventura (2012, 2016), Hirano and Yanagawa (2017) and Miao and Wang (2018). Other papers have shown that informational frictions and agency problems can give rise to bubbles in equilibrium (for example, Allen and Gorton, 1993, Allen and Gale, 2000, Allen, Barlevy and Gale, 2018; see Barlevy, 2018 for a survey of the literature).

Within the literature on rational asset price bubbles, the paper closest in spirit to ours is Miao-Wang. They show that a *stock* price bubble can arise in equilibrium in a production economy with infinitely-lived agents. In their model, borrowing is restricted by a firm's (market) value, and a bubble on the firm's stock relaxes the borrowing constraint, increasing the borrowing capacity of the economy.[9] The bubble exists in equilibrium as it provides a liquidity premium, which in turn yields a return on the bubble that is lower than the return on the

[8]Despite the predominant view that real economies are dynamically efficient, Geerolf (2013), more recently, has questioned the findings by Abel, Mankiw, Summers and Zeckhauser (1989).

[9]While the focus in our paper and in Miao-Wang is on a stock price bubble, many papers have focused on studying the existence of pure bubbles, like money, in production economies. Pure bubbles can also provide liquidity by raising borrowers' net worth (Caballero and Krishnamurthy, 2006; Kiyotaki and Moore, 2012; Aoki, Nakajima and Nikolov, 2014; Ikeda and Phan, forthcoming). However, the borrowing constraints in models of pure bubbles are different than ours and that in Miao-Wang because they do not depend on the stock market value of the firm.

stock, thereby satisfying the transversality condition. In our paper, we augment the model of Miao-Wang to allow for occasionally binding borrowing constraints.

There are a few other papers in this literature, more narrowly looking at the interaction of asset price bubbles and financial frictions, that are related to ours. Hirano and Yanagawa (2017) show that the effects of a bubble burst depend on the degree of pledgeability of the bubbly asset. They also show that bubbles can increase welfare, regardless of the effects of their bursts, as they relax financial frictions and allow for consumption smoothing by credit-constrained agents. On the other hand, Chauvin, Laibson and Mollerstrom (2011) find that in the absence of financial frictions bubbles are always welfare-reducing as they magnify cyclical fluctuations of consumption. Martin and Ventura (2016) propose a new rationale for macroprudential regulation: Borrowing should be taxed (or subsidized) such that it replicates the "optimal" bubble in the economy, maximizing output and consumption. Similarly to them, bubbles in our economy relax financial constraints and can be beneficial. Unlike them, the inefficiencies introduced by the bubble in our paper do not stem from crowding out resources away but from pecuniary externalities. Aoki and Nikolov (2015), Ikeda and Phan (2016), and Bengui and Phan (2018) study the effect of bubbles on risk-taking incentives and financial stability. Miao, Wang and Zhou (2015) find that loan-to-value limits and a property transaction tax can reduce the benefits of holding the bubbly asset, while Miao and Wang (2015) suggest that increasing bank capital requirements can mitigate the adverse consequences of bubbly bank valuations. Compared to our work, the last two papers consider ex-post policy interventions when borrowing constraints always bind; whereas we focus on ex-ante macroprudential regulation when constraints bind only occasionally. Moreover, policy analysis in those frameworks is based on a comparative statics exercise, whereas we study a fully-fledged Ramsey problem. Finally, we model a real economy and focus on financial regulation rather than monetary policy to tackle the evolution of the bubble; see Asriyan, Fornaro, Martin and Ventura (2020) and Gali (2014, 2021) for such models.

Our paper also contributes more broadly to the literature on optimal macroprudential policy. Macroprudential policy intervention is usually motivated by the presence of financial frictions, generating pecuniary externalities (Bianchi-Mendoza; Stein, 2012; Bianchi, 2011; Jeanne and Korinek, 2010; Davila and Korinek, 2018), or by the presence of aggregate demand externalities (Eggertsson and Krugman, 2012; Korinek and Simsek, 2016; Farhi and Werning, 2016). Within this area of research, the paper that is most closely related to ours is Bianchi-Mendoza. They also study the design of optimal macroprudential policy with commitment in a small open economy with occasionally binding borrowing constraints and pecuniary externalities, but they do not consider the effects of elevated asset prices on the optimal policy design. Another related paper is Biswas, Hanson and Phan (2020), who study the welfare effects of bubbles and subsequent policy intervention in an environment with downward wage rigidities and aggregate demand externalities. They find that policy should lean against bubbles because after a bubble's collapse the aggregate economic activity dips below the pre-bubble trend. Although their policy result is reminiscent of ours, a key difference—aside from the nature of the externalities—is that we find "leaning against bubbles" is only optimal when debt levels are high; otherwise, policy should be accommodative. Note that the richer policy prescription of our model stems from modeling the full non-linear dynamics of stochastic bubbles under occasionally binding borrowing constraints, while their analysis centers around "stationary" stochastic bubbles. As we explain in detail, this approach of modeling bubbles has been the norm in the literature because of the difficulties associated with a broader dynamic analysis that we attempt to tackle in our paper.

3 Model economy

We consider a small open economy with a rational bubble on a productive asset and an occasionally binding borrowing constraint. The modeling framework is very similar to Bianchi-Mendoza, but also features a rational asset price bubble. We model the bubble as in Miao-Wang, who show that a rational stock price bubble can be supported in equilibrium in production economies with infinitely-lived agents. We proceed by first outlining the competitive economy (CE) equilibrium allocations. Subsequently in section 4, we analyze the time-consistent optimal policy and derive the optimal macroprudential tax on borrowing that decentralizes the social planner's (SP) allocations.

3.1 Competitive economy with an asset price bubble

The economy is populated by a continuum of mass one of two types of infinitely-lived representative agents: households and firms. Households consume, provide labor services, and are the owners of firms. They can also frictionlessly trade firm shares in the stock market. Firms own a production technology, which combines capital, labor and intermediate goods as inputs to production. They purchase and sell capital, borrow internationally in the inter-temporal debt market, and do not own or trade the shares of other firms in the stock market.

Households. The representative household lives for infinite periods and maximizes the expected utility, which is a function of consumption, c_t, and labor, l_t,

$$\max_{c_t, l_t, \eta_{t+1}} E_t \sum_{t=0}^{\infty} \beta^t U\left(c_t - G(l_t)\right), \tag{1}$$

subject to the budget constraint,

$$c_t + \int_j \left(V_t^j - D_t^j\right) \eta_{t+1}^j dj = \int_j V_t^j \eta_t^j dj + w_t l_t, \tag{2}$$

where V_t^j denotes firm j's cum-dividend equity value, and D_t^j is firm j's dividend paid out in period t. The household starts the period with η_t^j shares of firm j, which it can trade in the stock market for shares of other firms. As a result, its end-period holdings of firm j's equity are η_{t+1}^j. In addition to dividends, the household earns income from wages, denoted by w_t, for labor supplied to firms. The utility function $U(\cdot)$ is a standard concave, twice-continuously differentiable in both its arguments and satisfies the Inada conditions. As in Bianchi-Mendoza, preferences are defined over a composite commodity $c_t - G(l_t)$, where $G(l_t)$ is a convex function, strictly increasing and continuously differentiable with $G(0) = 0$.[10]

The first-order optimality conditions of the household are as follows:

$$\lambda_t = U_{c,t}, \tag{3}$$

$$w_t = G_{l,t}, \tag{4}$$

$$V_t^j = \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} V_{t+1}^j \right] + D_t^j, \tag{5}$$

where λ_t is the Lagrange multiplier corresponding to the budget constraint (2), and $U_{c,t}$ and $G_{l,t}$ are the first-order derivatives of U and G with respect to c_t and l_t, respectively. Equation (3) denotes the marginal utility of the household with respect to consumption, and equation (4) denotes the household's optimal labor choice. The Euler equation (5) implies that a firm's equity is priced using the household's stochastic discount factor $\beta \frac{U_{c,t+1}}{U_{c,t}}$. The standard transversality condition is

$$\lim_{T \to \infty} \beta^T \frac{U_{c,T}}{U_{c,t}} V_T^j = 0, \tag{6}$$

where we have used the fact that $\eta_T^j = 1$ for all T and all j.

Firms. As all firms are the same in equilibrium, we explain the problem of a representative firm and drop the superscript j for notation simplicity. Denote by $V_t(k_t, L_t, b_t)$ the (stock) market value of a firm that enters period t with capital k_t and outstanding debt L_t issued at $t-1$, which is non-state contingent and matures at t. $V_t(k_t, L_t, b_t)$ also incorporates a bubble component b_t, so firms with the same level of capital and debt can have different market values because of bubbly valuations not tied to the fundamentals.

The firm's managers act in the best interest of shareholders (households) and choose new capital k_{t+1} and debt L_{t+1}, hire labor l_t^d, and purchase intermediate good v_t to maximize the market value of the firm. Hence, using (5), $V_t(k_t, L_t, b_t)$ should satisfy the following Bellman equation,

$$V_t(k_t, L_t, b_t) = \max_{\substack{k_{t+1}, L_{t+1}, \\ v_t, l_t^d}} D_t + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} V_{t+1}(k_{t+1}, L_{t+1}, b_{t+1}) \right], \tag{7}$$

where the firm's managers take the stochastic discount factor of household, $\beta U_{c,t+1}/U_{c,t}$, as given.

The dividends D_t are given by

$$D_t = y_t - p^v v_t - w_t l_t^d + \frac{L_{t+1}}{R} - L_t + q_t k_t - q_t k_{t+1}, \tag{8}$$

where $y_t = z_t F(k_t, l_t^d, v_t)$ is the total output at t given a Cobb-Douglas production function, $F(\cdot)$, which combines labor, l_t^d, with capital purchased in the previous period, k_t, and an intermediate good, v_t, which is traded in competitive world markets at a fixed exogenous price p^v; z_t is an aggregate productivity shock. R is the world-determined gross real interest rate taken

[10]The formulation of this composite commodity is defined by Greenwood, Hercowitz and Huffman (1988) and removes the wealth effect on labor supply inducing a countercyclical increase in the labor supply during crises.

as given in the small open economy with $\beta R < 1$.[11] The price of capital, q_t, is endogenously determined by equating supply and demand in the market for physical capital; while we assume a fixed supply of capital $K_t = 1$. Overall, dividends are equal to the output remaining after paying the factors of productions, v_t and l_t^d, the net capital expenditure, $q_t(k_{t+1} - k_t)$, and the net debt issuance, $L_{t+1}/R - L_t$.

We assume that a firm cannot raise equity and that its borrowing decision is limited by a borrowing constraint, which is *endogenously* derived from a limited commitment problem similar to Jermann and Quadrini (2012) and Bianchi-Mendoza (see section A.1 of the Appendix). The total liabilities of the firm at the beginning of the period comprise of $L_{t+1}/R + \theta p^v v_t$, which implies that, in addition to its inter-temporal borrowing, the firm also needs to finance ahead of production a portion $\theta \leq 1$ of the intermediate good purchases, $p^v v_t$. While L_{t+1} is an inter-temporal loan, $\theta p^v v_t$ is repaid within the same period and hence it does not bear any interest. Both types of borrowing can be diverted within period t, resulting in the following borrowing constraint:

$$\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t E_t \left[\beta \frac{U_{c,t+1}}{U_{c,t}} V_{t+1}(k_t, 0, b_{t+1}) \right]. \tag{9}$$

Constraint (9) limits the size of total borrowing to a fraction $m_t < 1$ of the firm's continuation market value (going-concern value), should the firm attempt to divert borrowed funds. In particular, creditors can detect if diversion takes place and lawfully enact loan covenants to seize the entire firm with probability m_t. If successful, they restructure the seized firm by cancelling its debt, but not liquidating its capital, and sell it as a whole within period t back to the households. Note that the restructured firm does not pay any dividends at t, thus households will value it only according to its future resale value at $t + 1$. In turn, the market value of the restructured firm with capital k_t and zero debt is given by $V_{t+1}(k_t, 0, b_{t+1})$ in each state at $t+1$, where b_{t+1} is the bubbly valuation at $t + 1$. The price that households would be willing to pay for the restructured firm at t is equal to its expected market price at $t + 1$ evaluated at the stochastic discount factor of households, i.e. $E_t[\beta U_{c,t+1}/U_{c.t} V_{t+1}(k_t, 0, b_{t+1})]$. As a result, the firm can borrow today against its expected going-concern value to creditors, which includes the discounted future cashflows from production as well as any additional valuation from future bubbles b_{t+1}. This form of the borrowing constraint ensures that a *stock price bubble* can be supported in equilibrium. Like in Miao-Wang, the bubble has a positive value in equilibrium because it can relax the borrowing constraint and allow the firm to borrow more, which in turn increases the firm's value supporting the initial (bubbly) valuation.[12]

Asset price bubble. To derive the value of the bubble, we solve the firm's dynamic programming problem (7) subject to (8) and (9), following the method of undetermined coefficients.

[11] Note that this formulation is equivalent to a closed economy with deep-pocketed, risk-neutral savers with time-discount factor $\beta' = 1/R > \beta$.

[12] For the bubble to be supported in equilibrium, it is crucial that the bubble is attached to the stock price of the firm and not to the liquidation value of the firm's physical capital. See Miao-Wang for details. As mentioned in the introduction, recent empirical evidence on corporate debt contracts (Lian and Ma, 2020, Kermani and Ma, 2020) supports defining firms' borrowing constraint as a function of the going-concern value.

Thus, we guess that the value function that the firm maximizes, takes the following form

$$V(k_t, L_t, b_t) = a_t k_t + s_t L_t + b_t, \tag{10}$$

where a_t and s_t are coefficients associated with the fundamentals of the model. The third coefficient, b_t, is not related to the firm's fundamentals and is interpreted as a bubbly component.

In section A.2 in the Appendix, we solve the dynamic programming problem of the firm and show that

$$a_t = [z_t F_{k,t} + q_t(1 + m_t \mu_t)], \tag{11}$$

$$s_t = -1, \tag{12}$$

$$b_t = (1 + m_t \mu_t)\beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1} \right], \tag{13}$$

where $F_{k,t} \equiv F(k_t, l_t^d, v_t) - F_{l,t} l_t^d - F_{v,t} v_t$, $F_{l,t}$ and $F_{v,t}$ are the marginal products of capital, labor and the intermediate good, respectively; and $\mu_t U_{c,t}$ is the Lagrange multiplier on the borrowing constraint (9). Condition (13) governs how the bubble accumulates over time.

After substituting (11) and (12) into (10), the value function takes the following form,

$$V_t(k_t, L_t, b_t) = [z_t F_{k,t} + q_t(1 + m_t \mu_t)]\, k_t - L_t + b_t. \tag{14}$$

Hence, the stock market value of the firm at t has the following components. First, it includes the cashflow value from production, $z_t F_{k,t} k_t$, using the existing capital, k_t. Second, it includes the value of the capital, $q_t k_t$, as well the shadow value of relaxing the borrowing constraint, $m_t \mu_t q_t k_t$. Third, the value of the firm declines with the outstanding debt level, L_t, and fourth, it increases with the bubbly valuation, b_t, that equity investors assign to the firm.

Then the going-concern value of the firm at t, which enters in the borrowing constraint (9), is equal to the market value of the restructured firm, $E_t[\beta U_{c,t+1}/U_{c,t} V_{t+1}(k_t, 0, b_{t+1})]$. Using the conjectured value function (14) and substituting the derived coefficients a_{t+1} and b_{t+1} using (11) and (13), the borrowing constraint (9) simplifies to

$$\begin{aligned} &\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t \left[\beta E_t \left(\frac{U_{c,t+1}}{U_{c,t}} a_{t+1} \right) k_t + \beta E_t \left(\frac{U_{c,t+1}}{U_{c,t}} b_{t+1} \right) \right] \\ \Rightarrow &\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t \left[q_t k_t + B_t \right], \end{aligned} \tag{15}$$

where we used the fact that $q_t = \beta E_t \left[U_{c,t+1}/U_{c,t} a_{t+1} \right]$ and where

$$B_t = \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1} \right]. \tag{16}$$

We will refer to equation (16) as the *collateral value of the bubble.*

From formulation (15), we can see that the collateral value of the firm as going-concern has a fundamental component and a bubbly component. The former is the value of the existing physical capital, $q_t k_t$, which incorporates the expected discounted value of all future cashflows from using the capital for production. The bubbly component is equal to the discounted expected value of future bubbles, b_{t+1}, rather than the current bubble value, b_t, because lenders care about the resale value of the restructured firm at the end of period t.

By combining (13) with (16), we can derive an equation that links the stock market value of the bubble, b_t, to the collateral value of the bubble, B_t

$$b_t = (1 + m_t \mu_t) B_t. \tag{17}$$

Equation (17) implies that the collateral value of the bubble, B_t, is less than the stock market value of the bubble, b_t, when the borrowing constraint binds in the present period. We will refer to equation (17) as the *bubble pledgeability condition* because it shows how the pledgeable portion of a stock market bubble is related to the tightness of the constraint, μ_t (see section 4 for a detailed discussion).

Competitive equilibrium. The following proposition outlines the representative firm's optimality conditions, while the proof is relegated to Appendix A.3.

Proposition 1. *The representative firm chooses $k_{t+1}, L_{t+1}, l_t^d, v_t$ to maximize its objective function* (7)*, given the functional form* (14)*, subject to the budget constraint* (8) *and the borrowing constraint* (15)*. In equilibrium, the optimality conditions (i)-(vi) below are satisfied:*
(i) the Euler equation with respect to borrowing, L_{t+1},

$$1 = \beta E_t \frac{U_{c,t+1}}{U_{c,t}} R + \mu_t, \tag{18}$$

ii) the Euler equation with respect to capital, k_{t+1},

$$q_t = \beta E_t \left\{ \frac{U_{c,t+1}}{U_{c,t}} [z_{t+1} F_{k,t+1} + q_{t+1}(1 + m_{t+1}\mu_{t+1})] \right\}, \tag{19}$$

iii) the labor, l_t^d, optimality condition,

$$w_t = z_t F_{l,t}, \tag{20}$$

iv) the intermediate good, v_t, optimality condition,

$$p^v (1 + \theta \mu_t) = z_t F_{v,t}, \tag{21}$$

v) the condition for how the bubble accumulates,

$$b_t = (1 + m_t \mu_t) \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1} \right], \tag{22}$$

vi) the bubble pledgeability condition,

$$b_t = (1 + m_t \mu_t) B_t, \tag{23}$$

vii) the complementarity slackness condition,

$$\mu_t \left[m_t(q_t k_t + B_t) - \frac{L_{t+1}}{R} - \theta p^v v_t \right] = 0. \tag{24}$$

Two observations follow from the firm's optimality conditions. First, the borrowing constraint (15) distorts both the optimal inter- and intra-temporal margins when binding. Condition (21), defining the choice of the intermediate good, embeds an additional cost, i.e. the cost of collateral financing equal to $\theta\mu_t p^v$. In addition, both Euler equations are distorted. The Euler equation for borrowing (18) implies that the marginal benefit from increasing borrowing today outweighs the expected future marginal cost by an amount equal to the shadow price of relaxing the borrowing constraint. Similarly, the Euler equation with respect to capital (19), equating the marginal cost of an extra unit of capital with its marginal benefit, embeds an additional benefit that derives from relaxing the borrowing constraint, valued at $m_{t+1}\mu_{t+1}q_{t+1}$. Second, not only the fundamental price of capital, but also the bubble depend on endogenous choice variables. This dependence generates pecuniary externalities operating separately via the two, which will be at the core of our policy analysis in section 4. Finally, note that we spell out conditions (v) and (vi), separately since the former governs the link between b_t and b_{t+1}, and the latter governs the link between b_t and B_t. As we will see in section 4.1, both of these linkages are at the heart of the optimal policy analysis.

Nature of the bubble. First, observe that a rational stock market bubble cannot exist in the deterministic steady state of the model, where $b_t = b_{t+1} = \overline{b}$ (or equivalently, $B_t = B_{t+1} = \overline{B}$), $q_t = q_{t+1} = \overline{q}$, and $U_{c,t+1} = U_{c,t} = \overline{U}_c$. For $\overline{b}, \overline{B} > 0$ and finite, conditions (19), (22), and (23) cannot hold together because that would require $z_{t+1}F_{k,t+1} \to 0$. Thus, the only deterministic steady state is a bubbleless steady state with $\overline{b} = \overline{B} = 0$. At the same time, from (18), it follows that $\overline{\mu} = 1 - \beta R > 0$, i.e. the borrowing constraint binds in the deterministic steady state. Yet, this is not enough to support a stock market bubble. Therefore, we focus on stochastic bubbles, i.e. bubbles that exist initially and may burst at each date with a positive probability after which they do not reappear (Blanchard and Watson, 1982; Weil, 1987).[13]

Denote by $b_0 > 0$ the initial level of the stock market bubble and by π the probability that it persists in each period thereafter (that is, it may burst with probability $1 - \pi$). We are agnostic about how the initial bubble is generated and focus on cases where the bubble appears at some point in time and persists thereafter. As such, we will treat the initial bubble emergence as a sunspot which coordinates agents beliefs on an equilibrium with a stochastic bubble. In other words, after it emerges the stochastic bubble needs to be endogenously priced and have a positive value in every period. This requires the growth rate of the bubble to satisfy equation (13). As we are interested in the interaction of the debt dynamics with stock market bubbles, we will consider non-stationary, stochastic bubbles and study the global dynamics of the bubble growth process. This is contrary to most of the literature, which typically focuses on stationary stochastic bubbles, i.e. bubbles that have a constant value in a stationary equilibrium before

[13] More recently, Guerron-Quintana, Hirano and Jinnai (2019) develop a model of recurrent bubbles in an environment with endogenous growth and infinitely-lived households.

they burst (see, for example, Kocherlakota, 2009, Miao and Wang, 2015,, Miao and Wang, 2018, Biswas, Hanson, and Phan, 2020).[14]

Finally, the bubble we consider is attached to the stock price of the firm. In our model "pure bubbles", i.e. bubbles on otherwise useless assets, do not exist, but the model can be easily extended by introducing an additional intrinsically useless asset, akin to real estate in Miao, Wang and Zhou (2015), which can have a positive valuation as long as it serves as collateral. As a result, we would be able to study the differential liquidity properties of stock price bubbles and bubbles attached to other assets used as collateral.

The competitive equilibrium of the economy is defined as follows.

Definition of equilibrium. *For given initial values of L_0 and b_0, probability π, and exogenous processes $\{z_t, m_t\}_{t=1}^{\infty}$, a competitive equilibrium for the economy with a stochastic stock market bubble and a borrowing constraint is a sequence of allocations $\left\{c_t, l_t, l_t^d, v_t, y_t\right\}_{t=0}^{\infty}$, an asset profile $\{k_{t+1}, L_{t+1}\}_{t=0}^{\infty}$, bubble processes $\{b_t\}_{t=1}^{\infty}$ and $\{B_t\}_{t=0}^{\infty}$, and a price system $\{q_t, w_t, p^v, R\}_{t=0}^{\infty}$ such that:*

1. Given the price system $\{q_t, w_t, p^v, R\}_{t=0}^{\infty}$ and bubble processes $\{b_t\}_{t=1}^{\infty}$ and $\{B_t\}_{t=0}^{\infty}$, the allocations and the asset profile solve households' and firms' problems, i.e. conditions (3)-(5) *and* (18)-(23) *are satisfied,*

2. The markets for labor, capital, and equity clear, $l_t^d = l_t$, $k_t = K_t = 1$, $\eta_t^j = 1\ \forall \mathrm{j}, \mathrm{t}$, and

3. The resource constraint holds, $c_t + L_t = L_{t+1}/R + z_t F\left(1, l_t, v_t\right) - p^v v_t$.

Bubble existence. We now show that a non-stationary, stochastic rational bubble can be indeed supported in equilibrium if two conditions are satisfied.[15] First, the bubble has to be priced in equilibrium to ensure rationality, i.e. (22) needs to hold for every t. Second, the transversality condition (6) needs to be satisfied. In Miao-Wang, firms are homogeneous in the beginning of every period but are hit by idiosyncratic shocks such that the constraint binds only for some of them. Yet there is always a positive probability that the constraint will bind for the ex ante homogeneous firm, which is sufficient to generate a positive liquidity premium in each period. As a result, it is straightforward to show that the transversality condition cannot exclude the bubble. In our model, there is no ex post firm heterogeneity and the borrowing constraint binds only occasionally, that is $\mu_t > 0$ for some t and $\mu_t = 0$ for other t. Thus, it is not obvious that the transversality condition will not exclude a rational bubble that satisfies (22).

To prove that rational bubbles can exist in our setup, we consider the asymptotic behavior of the economy. Intuitively, a rational bubble can be supported even if the liquidity premium is only occasionally positive as long as the borrowing constraint binds with a positive probability as t goes to infinity. In other words, a rational bubble can exist if it continues to be useful to mitigate financial frictions. The following proposition establishes that this is true for sufficiently low values of the parameter m_t. The proof relies on the fact that in incomplete markets

[14]Given that the borrowing constraint may only occasionally bind in our model, equations (22) and (23) imply that we cannot obtain a stationary stochastic bubble.

[15]Miao-Wang show that both deterministic (steady state) and stochastic (stationary) stock market bubbles can exist in their framework, but do not study non-stationary stochastic bubbles. Interestingly, the condition under which a deterministic bubble exists excludes the existence of a stationary stochastic bubble, and vice versa.

economies, the natural debt limit will be binding infinitely often (see Aiyagari, 1994), and that the borrowing constraint will be more restrictive than the natural debt limit if the pledgeable portion of the representative firm's value is low enough.

Proposition 2. *If* $\min(m_t) \leq \overline{m}$*, the borrowing constraint (15) binds infinitely often. That is,* $lim_{t\to\infty} \sup \mu_t > 0$*, and a rational bubble can be supported in equilibrium.*

Equilibrium solution. The initial value of the stock market bubble, b_0, is determined exogenously by a sunspot realization. Thus, for an initial level of debt L_0, the competitive equilibrium is characterized by choices of new borrowing and capital, the amount of the intermediate good and labor inputs, prices for capital and labor, current consumption and production, as well as the collateral value of the bubble. Regarding the latter, B_0 is endogenous and does not only depend on b_0, but also on how binding the borrowing constraint is, through condition (23). As a result, $B_0 \leq b_0$, depending on whether the borrowing constraint binds or not. Thereafter, the stock market bubble grows endogenously. This structure suggests that not only the level of debt, but also the level of the stock market bubble, matter for all other choices. Consider for example some L_0 and two alternative sunspot realizations b_0' and b_0'': The two equilibrium paths will not be the same. Thus, we need to solve for equilibrium allocations corresponding not only to all possible values for L_0, but also for all possible realizations of b_0.[16]

To find the equilibrium, we need to compute stationary policy functions $x_t \equiv x(L_t, z_t, m_t, b_t)$, where $x_t \in \{c_t, v_t, l_t, L_{t+1}, \mu_t, w_t, q_t, B_t\}$. We need a law of motion for the endogenous bubble state in addition to the (usual) law of motion for the endogenous debt state and distributions for the other exogenous state variables. The only equilibrium requirement is that the bubble growth satisfies (22), which only needs to hold in expectation and, thus, can encompass various bubble processes. We restrict our attention to beliefs about a bubble process that imply $b_{t+1} = b_t U_{c,t}/[\beta(1 + m_t\mu_t)\pi U_{c,t+1}]$.[17] This condition cannot yet serve as the law of motion for how the bubble grows from b_t to b_{t+1} because the right-hand side depends not only on pre-determined variables, but also on con-current choice variables through $U_{c,t+1}$ and, hence, on b_{t+1} itself. We follow an equilibrium selection procedure that replaces b_{t+1} with a future state b^*_{t+1} that only depends on the other current and future state variables, but not on $t+1$ choice variables, while imposing consistency of expectations.[18] In particular, b^*_{t+1} is the solution (fixed-point) to

$$b^*_{t+1} = \frac{b_t U_{c,t}}{\beta(1 + m_t\mu_t)\pi U^*_{c,t+1}}, \tag{25}$$

where $U^*_{c,t+1} = U'(x(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}))$. In short, among all possible future bubble states at $t + 1$, b_{t+1}, we select the one, b^*_{t+1}, that satisfies how rational agents price it in equilibrium.

Then, the law of motion for the bubble is given by the function $b^*(L_{t+1}, z_{t+1}, m_{t+1}, L_t, z_t, m_t)$

[16] The level of existing capital k_t is also a state variable, but it does not matter for the equilibrium outcomes because $k_t = K_t = 1$ for every t.

[17] These beliefs are consistent with (22) and allow for state-contingent growth of the bubble. We abstract from additional exogenous shocks to the realized bubble value at $t + 1$ as in Martin and Venture (2012, 2016) .

[18] Solving the model is easier in case of stationary bubbles, or non-stationary bubbles and linear preferences (e.g. Farhi and Tirole (2012*a*)) because in both cases $U_{c,t+1}/U_{c,t} = 1$. However, as mentioned, studying the global dynamics of indebtedness and bubbly valuations requires the modeling of non-stationary bubbles, while concave utilities are needed to generate pecuniary externalities and a role for policy.

that yields b^*_{t+1} given other state variables, but not future choice variables. Section A.5 in the Appendix presents the detailed steps of the equilibrium selection procedure. Thus, we replace the equilibrium condition (22) with the stricter condition (25) that implicitly defines the law of motion $b^*(L_{t+1}, z_{t+1}, m_{t+1}, L_t, z_t, m_t)$. We will call condition (25) the *bubble accumulation condition* because it shows how the current bubble state, b_t, is expected to grow to the future bubble state b^*_{t+1}.

The expectation terms in (18) and (19) also depend on the $t+1$ bubble state. Using (25), we can re-write these equilibrium conditions in recursive form in terms of b^*_{t+1} as

$$U_{c,t}(1-\mu_t) = \beta R E_t \mathcal{H}_u(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t), \tag{26}$$

$$q_t U_{c,t} = \beta E_t \mathcal{H}_q(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t), \tag{27}$$

where the policy functions $\mathcal{H}_u$ and $\mathcal{H}_q$ give the $t+1$ values in the right-hand side of (18) and (19) when the future bubble state is (selected to be) b^*_{t+1} (equations (A.26) and (A.27) in the Appendix). We are now equipped to express the competitive equilibrium recursively.

Recursive representation of competitive equilibrium. Define the state space at t as (L_t, m_t, z_t, b_t) and the state space at $t+1$ as $(L_{t+1}, m_{t+1}, z_{t+1}, b^*_{t+1})$. The competitive equilibrium is characterized by: (i) policy functions $x(L_t, m_t, z_t, b_t)$ for period-t endogenous variables $x_t \in \{c_t, v_t, l_t, L_{t+1}, \mu_t, w_t, q_t, B_t\}$; (ii) policy function $\mathcal{H}_u$ and $\mathcal{H}_q$ for the forward looking terms in (A.26) and (A.27); and (iii) values b^*_{t+1} that the bubble b_t grows in each future state, such that all markets clear, the resource constraint holds, and the optimality conditions (3)-(5), as well as (20), (21), (23), (25) (26), and (27) are satisfied.

Expressing the equilibrium recursively is not only important to study the non-monotonic dynamics of indebtedness and bubbly valuations, but is also needed to derive the optimal time-consistent policy in the next section.

4 Optimal macroprudential policy

To derive the optimal policy, we proceed by first formulating the social planner's problem and then discussing the properties of the optimal taxation that implements the planner's solution.

4.1 Time-consistent planner's problem

The policy design follows the Ramsey approach, which consists of the social planner choosing policies, prices, and allocations in order to maximize the economy's social welfare function. In doing so, the planner has to respect all equilibrium conditions of the recursive representation of the competitive equilibrium described in the previous section, apart from those that the planner explicitly distorts with policy tools. This ensures that the allocations chosen by the planner can be implemented as allocations in the competitive economy.

Unlike the standard Ramsey literature, where the planner optimally chooses distortionary policies intended to finance government expenditure, the planner in our model chooses policy to alleviate the externalities arising from agents' atomistic behavior. We assume that the only policy available to the planner is a state-contingent tax on borrowing, τ_t.[19] This instrument is Pigouvian in nature with the tax revenues being rebated lump-sum back to the private agents, with T_t denoting the transfer.

The resource constraint of the decentralized economy then takes the following form

$$c_t + L_t(1+\tau_{t-1}) + p^v v_t \leq z_t F(1, l_t, v_t) + \frac{L_{t+1}}{R} + T_t, \tag{28}$$

where $T_t = \tau_{t-1}L_t$. Equation (28) is obtained by adding the budget constraints of the two agents, (2) and (8), with the borrowing tax introduced, and using the market clearing conditions $l_t^d = l_t$, $k_t = 1$ and $\eta_t^j = 1$ for all t and j.

The Euler equation with respect to borrowing, (18), from the perspective of atomistic agents, who take $T_t = \tau_{t-1}L_t$ as given, then becomes

$$\begin{aligned} &U_{c,t}(1-\mu_t) = \beta R(1+\tau_t)E_t U_{c,t+1} \\ \Rightarrow &U_{c,t}(1-\mu_t) = \beta R(1+\tau_t)E_t \mathcal{H}_u(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t), \end{aligned} \tag{29}$$

using the recursive representation in (26). Note that this condition does not enter as a constraint in the planner's problem as the planner can implement the desired allocations in a competitive equilibrium by appropriately choosing the level of τ_t.

Moreover, we assume that the planner does not have the technology to commit to future policies.[20] Therefore, we solve for the optimal time-consistent macroprudential policy, taking into account the effects of the planner's current period choices on future planners' optimization problems. As a result, the planner does not have an incentive to deviate from policy rules of previous social planners.

The planner's maximization problem, at state (L_t, z_t, m_t, b_t), is given by

$$\max_{c_t, q_t, L_{t+1}, v_t, l_t} E_t \sum_{t=0}^{\infty} \beta^t U(c_t - G(l_t))$$

$$s.t \quad c_t + L_t + p^v v_t \leq z_t F(1, l_t, v_t) + \frac{L_{t+1}}{R} \quad (\lambda_t^p) \tag{30}$$

$$\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t \left(q_t + \frac{b_t}{1 + m_t \mu(l_t, v_t)} \right) \quad (\mu_t^p) \tag{31}$$

$$q_t U_{c,t} = \beta E_t \mathcal{H}_q(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}(c_t, \mu(l_t, v_t)), L_t, z_t, m_t, b_t) \quad (\xi_t) \tag{32}$$

$$z_t F_{l,t} = G_{l,t}, \quad (\zeta_t) \tag{33}$$

where the Lagrange multipliers associated with each constraint are given in parentheses. Note that we distinguish between the Lagrange multipliers on the budget (resource) and borrowing

[19] Alternative instruments that affect the inter-temporal margin, such as debt limits, loan-to-value ratios, can be used instead of the tax. The tax can also be imposed on the interest rate expenses.

[20] Bianchi-Mendoza show that the optimal policy under commitment is time-inconsistent since asset prices are determined by a dynamic condition linking the present and future (expected) marginal utilities of consumption. Instead, they follow the time-consistent approach under which a planner cannot commit at t to the whole path of future policy choices.

constraints in the competitive and in the planner's problem (λ_t and λ_t^p; μ_t and μ_t^p, respectively).

The constraints in the planner's problem can be explained as follows. First, the planner respects the resource constraint (28) in the decentralized economy, while taking into account that $T_t = \tau_{t-1}L_t$, reducing it to (30). Second, the planner is constrained by the same financial frictions as the private agents and, hence, needs to respect the borrowing constraint (31). The planner needs to internalize how the current production decisions affect the collateral value of the bubble, B_t, via the bubble pledgeability condition (23), i.e. $B_t = b_t/(1 + m_t\mu(l_t, v_t))$, which we have substituted in the right-hand side of the borrowing constraint (31)—recall that the stock market value of the bubble, b_t, is a state variable, but the collateral value of the bubble, B_t, is endogenously determined in each state. Third, the planner internalizes how the fundamental price of capital depends on consumption and production decisions, so we include the Euler condition with respect to capital, (27), as an additional constraint. In this constraint, we have also incorporated the bubble accumulation condition (25), yielding (32), in order to capture the dependence of the future bubble state, $b_{t+1}^*(c_t, \mu(l_t, v_t))$, on the current consumption and production decisions. This is important because, as mentioned, time-consistent policy should take into account the effects of the planner's current period choices on future planners' optimization problems. Fourth, constraint (33)—derived by combining (4) and (20)—governs the equilibrium in the labor market in the competitive economy, and guarantees that the planner's labor choice is implementable.[21] Finally, the planner internalizes how the tightness of the borrowing constraint in the competitive equilibrium, μ_t, is determined in order to guarantee that the planning allocations are implementable as competitive equilibrium outcomes. Hence, using (21), which governs the demand for the intermediate good in the competitive economy, the Lagrange multiplier on the collateral constraint in the competitive economy can be expressed as $\mu(l_t, v_t) = (z_t F_{v,t} - p^v)/(\theta p^v)$; we use this expression to substitute for μ_t where it appears in the planner's problem in order to highlight its dependence on the factors of production, l_t and v_t. Apart from these constraints, the planner's problem needs to satisfy the complementary slackness condition $[L_{t+1}/R + \theta p^v v_t - m_t(q_t + b_t/(1 + m_t\mu(l_t, v_t)))]\mu_t^p = 0$.

Clearly the bubble pledgeability and bubble accumulation conditions substituted in the planner's problem above are not relevant in a bubbleless economy as $b_t = 0$ and $b_{t+1}^* = 0$. In this case, (31) and (32) coincide with the ones in the (bubbleless) BM economy. Constraint (33) is also redundant in the planner's problem absent a bubble.[22] The intuition why this constraint is relevant only in the presence of a bubble is as follows: The tightness of the borrowing constraint in the decentralized competitive economy, measured by $\mu(l_t, v_t)$, matters for the size of the collateral value of the bubble. The smaller the Lagrange multiplier is, the bigger is the pledgeable portion of the stock market bubble and the bigger is the effect of the bubble on attenuating the financial frictions. At the same time, the planner's choices of the intermediate good and labor matter for the tightness of the borrowing constraint, which is the reason why (33) enters as an additional constraint. On the contrary, in the bubbleless economy, only the

[21] Given the equilibrium level for labor determined by (33), the equilibrium wage is given by either (4) or (20). The condition determining the wage is not an additional constraint for the planner, because wages do not generate any income effect as firms are owned by households and dividends are distributed in the period they accrue, as well as because payroll costs do not enter the borrowing constraint, following Bianchi-Mendoza.

[22] This can easily be shown by setting $b_t = b_{t+1}^* = 0$ in equations (A.37) and (A.39) in section A.6 in the Appendix.

fundamental price of capital matters for the borrowing constraint and, as can be seen from (32), it does not depend on the current value of $\mu(l_t, v_t)$. Hence, the planner can choose the level of the intermediate good and labor that satisfy the competitive equilibrium conditions without worrying how they matter for the tightness of the borrowing constraint, $\mu(l_t, v_t)$, in the bubbleless economy.

In section A.6 in the Appendix, we derive the optimality conditions that characterize the planner's problem. Herein, we focus on the main conditions to derive intuition about the mechanism through which the externalities operate. First, consider the optimality condition with respect to consumption, which can be expressed concisely as

$$\lambda_t^p = \underbrace{U_{c,t}}_{\substack{\text{Competitive}\\ \text{equilibrium}}} + \overbrace{\mathcal{F}_t q_t}^{\substack{\text{Fundamental-price}\\ \text{externality}}} + \underbrace{\mathcal{V}_t \beta E_t \frac{d\mathcal{H}_{q,t+1}}{db^*_{t+1}} \frac{db^*_{t+1}}{dc_t}}_{\substack{\text{Bubble-accumulation}\\ \text{externality}}} + \overbrace{\mathcal{W}_t \frac{dB_t}{dl_t}}^{\substack{\text{Bubble-pledgeability}\\ \text{externality}}}, \tag{34}$$

where $\mathcal{F}_t$, $\mathcal{W}_t$, and $\mathcal{V}_t$ are given by (A.41), (A.42), and (A.43) in the Appendix.

We begin our analysis by comparing the planner's first-order condition with respect to consumption, (34), to the corresponding condition of the competitive economy, $\lambda_t = U_{c,t}$, which equates the shadow value of income to the marginal utility. Compared to the competitive economy, the planner's optimality condition contains three additional terms.

The first additional term (second term in the right-hand of (34)) is unrelated to the bubble's presence and is positive (see section A.6 in the Appendix). This term captures how the planner—unlike the private agents—internalizes that an additional unit of consumption reduces today's marginal utility, putting upward pressure on the fundamental price via condition (32). A higher q_t relaxes the borrowing constraint enabling an additional increase in c_t and q_t, and a further relaxation of the constraint. This is the familiar externality highlighted by Bianchi-Mendoza, capturing the additional positive effects of higher consumption and making the shadow value of income for the planner higher than the marginal utility of consumption. We will refer to this externality as the *fundamental-price externality.*

The second and third additional terms in (34) are novel. The former, $\mathcal{V}_t \beta E_t d\mathcal{H}_{q,t+1}/db^*_{t+1} \cdot db^*_{t+1}/dc_t$, captures how current consumption affects the bubble growth rate from t to $t+1$, i.e. db^*_{t+1}/dc_t. Note that, for a given bubble state, b_t, the bubble grows endogenously to b^*_{t+1}, while satisfying condition (25). Through this equation, the current level of consumption affects the growth rate of the bubble. In turn, the level of the future bubble state, b^*_{t+1}, matters for q_t because it is incorporated in $\mathcal{H}_{q,t+1}$ in the right-hand side of (32). The planner takes the dependence of q_t on b^*_{t+1} and, thus, c_t into account, which is a link that private agents fail to internalize when making optimal decisions. We refer to the externality captured by this term as the *bubble-accumulation externality.* Both the fundamental-price and the bubble-accumulation externalities operate through how c_t affects q_t. But while the former works directly through the stochastic discount factor used to price q_t, the latter works indirectly through the growth rate for the bubble state.

The third additional term, $\mathcal{W}_t dB_t/dl_t$ captures how production decisions affect the collateral value of the bubble, B_t, as indicated by dB_t/dl_t, and the fact that $\mathcal{W}_t$ depends on the utilization of labor and the intermediate good (see condition (A.42) in the Appendix). Private agents fail to internalize how their production decisions affect the tightness of the borrowing constraint, $\mu(l_t, v_t)$, which in turn affects the size of the bubble that can be pledged as collateral, i.e. B_t, via equation (23). Unlike the private agents, the planner takes this dependence into account. We refer to the externality captured by this term as the *bubble-pledgeability externality*, which is positive as we show in the Appendix. Note that, although this externality operates via the factors of production, it still enters the consumption decision. This is because it affects the tightness of the constraint, and thus the strength of the fundamental price externality, which depends on c_t. As we show in the Appendix, $\mathcal{W}_t dB_t/dl_t$ can be alternatively written as

$$\underbrace{\mathcal{F}_t q_t}_{\substack{\text{Fundamental-price} \\ \text{externality}}} \quad \cdot \quad \underbrace{\mathcal{W}'_t \cdot \frac{dB_t}{dl_t}}_{\substack{\text{Bubble-pledgeability} \\ \text{amplification}}} \quad , \tag{35}$$

i.e. as a multiple of the fundamental-price externality. Since $\mathcal{W}' < 0$, given in equation (A.44), and $dB_t/dl_t < 0$, the bubble-pledgeability externality scales up the fundamental-price externality.

Clearly, the bubble-accumulation and bubble-pledgeability externalities are absent for $b_t = 0$. But, also, all three externality-terms in (34) are zero if the borrowing constraint does not bind.

Next, we discuss the Euler condition with respect to borrowing, which is a key component to derive the optimal policy, and takes the following form (by substituting (A.29) in (A.30))

$$\lambda_t^p = \beta R E_t \left(\lambda_{t+1}^p + \frac{m_t \mu_t^p}{U_{c,t}} \frac{d\mathcal{H}_{q,t+1}}{dL_{t+1}} \right) + \mu_t^p, \tag{36}$$

where $d\mathcal{H}_{q,t+1}/dL_{t+1}$ captures the fact that L_{t+1} is a state at $t+1$ and, thus, its choice at t matters for the determination of q_t through the forward-looking terms in (32).

Compared to the corresponding optimality condition (18) for L_{t+1} of the private agents, the planner's optimality condition (36) incorporates the three aforementioned externalities. We discuss separately the cases that the borrowing constraint binds only occasionally at t.

1. The borrowing constraint does not bind at t, but may bind at $t+1$: In this case, $\mu_t^p = \mu_t = \mathcal{F}_t = \mathcal{W}_t = \mathcal{V}_t = 0$ and (36) becomes $U_{c,t} = \beta R E_t \lambda_{t+1}^p$. Hence, the marginal cost of borrowing at t differs for the social planner and the private agents as the former accounts for the three externalities described above. The exact amount of the difference is equal to $\beta R E_t(\lambda_{t+1}^p - U_{c,t+1})$, where λ_{t+1}^p is obtained from (34), evaluated at period $t+1$. By taking into account the externality terms in the the Euler equation with respect to borrowing, the planner internalizes that higher debt at t curtails the future borrowing capacity and, thus, consumption when the constraint binds at $t+1$. Lower c_{t+1} puts pressure on q_{t+1}, which further tightens the constraint. Moreover, a tighter borrowing constraint decreases the collateral value of the bubble via the bubble-pledgeability externality. Finally, lower c_{t+1} decreases the rate of growth for the bubble from $t+1$ to $t+2$ and results in a lower b_{t+2}^*, which matters for q_{t+1} via the bubble-accumulation externality. Contrary to the previous two externalities, we cannot unambiguously say whether the latter externality adds on the planner's marginal cost of borrowing or not.

2. The borrowing constraint binds at t and may bind at $t+1$: In the event the borrowing constraint binds at t and may also bind at $t+1$, the three externalities described above are present in both periods, pushing the marginal cost of borrowing from the perspective of the planner in opposite directions. In addition, the planner also takes into account how the choice of L_{t+1} affects the level of q_t via the forward-looking expectations captured in $E_t H_{q,t+1}$. Intuitively, the planner faces a trade-off between choosing allocations such that it increases current valuations, q_t and B_t, at the cost of potentially decreasing future valuations, q_{t+1} and B_{t+1}. The overall effect on the marginal cost of borrowing compared to the competitive economy depends on which of the two opposing forces dominates.

4.2 Optimal tax rate in the presence of a bubble

The optimal tax on borrowing can be derived by combining the Euler equation for L_{t+1} of the planner (36) with the corresponding equation of the agents incorporating the tax on borrowing (29), which yields

$$\tau_t = \underbrace{-\frac{\lambda_t^p - U_{c,t}}{\beta R E_t U_{c,t+1}} + \frac{\mu_t^p - U_{c,t}\mu(l_t, v_t)}{\beta R E_t U_{c,t+1}} + \frac{m_t \mu_t^p}{U_{c,t}} \frac{E_t d\mathcal{H}_{q,t+1}/dL_{t+1}}{E_t U_{c,t+1}}}_{\substack{\text{Targeting current} \\ \text{valuation externalities}}} + \underbrace{\frac{E_t(\lambda_{t+1}^p - U_{c,t+1})}{E_t U_{c,t+1}}}_{\substack{\text{Targeting future} \\ \text{valuation externalities}}}. \quad (37)$$

The tax in (37) balances the effects of borrowing on the *current* versus the *future* valuation externalities described earlier. In order to focus on the macroprudential component of the borrowing tax, we consider that $\mu_t = \mu_t^P = 0$ (but $E_t \mu_{t+1}^p > 0$). The tax rate is given a macroprudential interpretation because it intends to hamper excessive borrowing in good times (when the borrowing constraint does not bind) to lower the risk of future instability due to deflating—fundamental and bubbly—valuations in bad times (when the borrowing constraint starts to bind). Using (34) and (37), the macroprudential tax is given by

$$\tau_t^{mp} = \frac{E_t \mathcal{F}_{t+1} q_{t+1}}{E_t U_{c,t+1}} + \frac{E_t \mathcal{W}_{t+1} \frac{dB_{t+1}}{dl_{t+1}}}{E_t U_{c,t+1}} + \frac{\beta E_t \mathcal{V}_{t+1} \frac{d\mathcal{H}_{q,t+2}}{db_{t+2}^*} \frac{db_{t+2}^*}{dc_{t+1}}}{E_t U_{c,t+1}}, \quad (38)$$

where $\mathcal{F}_{t+1}$, $\mathcal{W}_{t+1}$, $\mathcal{V}_{t+1}$, and db_{t+2}^*/dc_{t+1} are given by (A.41), (A.42), (A.43), and (A.33) applied to period $t+1$, respectively.

The first component in the macroprudential tax (38) addresses the fundamental-price externality operating via the q_{t+1}, which calls for a positive tax to lean against imbalances. The second component addresses the bubble-pledgeability externality operating via B_{t+1}, which as mentioned earlier amplifies the fundamental-price externality described in Bianchi-Mendoza and requires a more aggressive macroprudential response. The third component addresses the bubble-accumulation externality operating via b_{t+2}^*, but the direction at which it pushes the macroprudential tax is ambiguous.

We now compare the macroprudential tax τ_t^{mp} in the presence of the bubble to the total tax that would prevail absent a bubble, denoted by $\tilde{\tau}_t$. The reason why we compare τ_t^{mp} to the general tax in the bubbleless economy is that the constraint may bind at t only absent a bubble for certain state realizations. Thus, doing so, allows us to characterize how the bubble changes the optimal policy for the whole range of states that we obtain a macroprudential tax in the presence of a bubble. Note that $\tilde{\tau}_t$ is the optimal tax in the bubbleless BM economy, which can be obtained using (34) and (37) after setting $b_t = b_{t+1} = 0$ and $\mu_t^P = \lambda_t^P \mu(l_t, v_t)$ (see Appendix A.6):

$$\tilde{\tau}_t = \underbrace{-\frac{\tilde{\mathcal{F}}_t \tilde{q}_t (1+\mu(\tilde{l}_t, \tilde{v}_t))}{\beta R E_t \tilde{U}_{c,t+1}} - \frac{m_t(\tilde{U}_{c,t} + \tilde{\mathcal{F}}_t \tilde{q}_t)\mu(\tilde{l}_t, \tilde{v}_t)}{\tilde{U}_{c,t}} \frac{E_t d\tilde{\mathcal{H}}_{q,t+1}/d\tilde{L}_{t+1}}{E_t \tilde{U}_{c,t+1}}}_{\substack{\text{Tackling externalities} \\ \text{at } t}} + \underbrace{\frac{E_t \tilde{\mathcal{F}}_{t+1} \tilde{q}_{t+1}}{E_t \tilde{U}_{c,t+1}}}_{\substack{\text{Tackling externalities} \\ \text{at } t+1}}, \quad (39)$$

where the variables denoted by a tilde sign correspond to a bubbleless economy.

Then, we can express the difference between the macroprudential tax in the bubbly economy and the total tax in the bubbleless economy as

$$\begin{aligned} \tau_t^{mp} - \tilde{\tau}_t = & \overbrace{\frac{\tilde{\mathcal{F}}_t \tilde{q}_t (1+\mu(\tilde{l}_t, \tilde{v}_t))}{\beta R E_t \tilde{U}_{c,t+1}} + \frac{m_t(\tilde{U}_{c,t} + \tilde{\mathcal{F}}_t \tilde{q}_t)\mu(\tilde{l}_t, \tilde{v}_t)}{\tilde{U}_{c,t}} \frac{E_t d\tilde{\mathcal{H}}_{q,t+1}/d\tilde{L}_{t+1}}{E_t \tilde{U}_{c,t+1}}}^{\text{Current extensive margin}} \\ & \underbrace{+\frac{E_t \mathcal{F}_{t+1} q_{t+1}}{E_t U_{c,t+1}} - \frac{E_t \tilde{\mathcal{F}}_{t+1} \tilde{q}_{t+1}}{E_t \tilde{U}_{c,t+1}}}_{\text{Future extensive margin}} \underbrace{+\frac{E_t W_{t+1} \frac{dB_{t+1}}{dl_{t+1}}}{E_t U_{c,t+1}} + \frac{\beta E_t \mathcal{V}_{t+1} \frac{d\mathcal{H}_{q,t+2}}{db^*_{t+2}} \frac{db^*_{t+2}}{dc_{t+1}}}{E_t U_{c,t+1}}}_{\text{Intensive margin}}. \end{aligned} \quad (40)$$

The first component in (40) captures the fact that the borrowing constraint may bind in the bubbleless economy for some states z_t, m_t, L_t, but does not necessarily bind in the bubbly economy. Then, the planner in the bubbleless economy needs to account for the pecuniary externality that the binding constraint at t generates, which should call for reducing the tax on borrowing so as to support the current price $\tilde{q}_t$. In contrast, the presence of the bubble at t can directly relax the binding constraint and, thus, the planner does not need to be concerned about pecuniary externalities at t in the bubbly economy. We refer to this channel via which the bubble relaxes the current borrowing constraint and removes the fundamental-price externality at t as the *current extensive margin* of optimal policy.

The second component in (40) captures the part of the tax rate that tackles the expected (rather than the current) fundamental-price externalities when the constraints bind at $t+1$ in the bubbleless and the bubbly economy. The expectation operator (in the first ratio) encompasses both the states where the bubble persists with probability π and the states where it bursts with probability $1-\pi$. On the one hand, the mere presence of the bubble can make the borrowing constraint slack in some future states as long as the bubble does not burst, which pushes down the (conditional) probability of the constraint binding and calls for a lower tax. On the other hand, the equilibrium values of capital, consumption, and other variables also change in the presence of bubble. Thus, the pecuniary externalities operating via the value of capital may become exacerbated when the bubble bursts. This is so because the presence of the bubble in the previous periods may have allowed for more debt accumulation compared to the bubbleless case. We refer to the overall effect through which the macroprudential tax tackles the future fundamental-price externalities as the *future extensive margin* of optimal policy.

Both the current and future extensive margins accrue from the fact that the mere presence of the bubble can make, otherwise binding, borrowing constraints slack. Our results on the extensive margin should carry over to other models that feature bubbles as exogenous shocks. However, the collateral value of the bubble as well as the growth rate of the bubble are endogenous in our model and depend on current production and consumption decisions as explained earlier. This dependence introduces an additional margin captured by the third component in (40), which we refer to as the *intensive margin* of optimal policy. This part of the tax rate tackles jointly the bubble-pledgeability and the bubble-accumulation externality when the constraint binds at $t+1$. We should note that the intensive margin is present as long as the bubble does not burst at $t+1$ and, thus, it is not meant to tackle the adverse impact of the bubble bursting in the future (captured in the future extensive margin). Instead, it tackles the adverse externalities from deflating, but positive, bubbly valuations. The planner cannot affect the probability that the bubble persists, as it is exogenously driven by a change in sentiments, but can affect the size of the bubble to make sure that it maintains value exactly when it is most needed, i.e. when borrowing constraints start binding in the future. This distinguishes the intensive margin from the adverse effects of a bursting bubble studied in other papers.[23]

Since the sign of the combined effect of the two extensive and the intensive margins is ambiguous, in the next section we solve a calibrated version of the model numerically to examine which force, and under what conditions, prevails.

Finally, before turning to the quantitative analysis, it is worth mentioning that the macroprudential tax, τ_t^{mp}, in a bubbly economy is different than the macroprudential tax in a bubbleless economy with simply higher fundamental prices, i.e. a bubbleless economy with $q'_{t+1} = q_{t+1} + B_{t+1}$; but where debt levels, consumption, and other endogenous variables are the same.[24] In other words, the bubbly and bubbleless (with higher asset prices) economies do not feature an equivalent macroprudential policy. Clearly, the functional form of the bubble-accumulation externality is unique to the bubbly economy since it captures how $t+1$ choices affect the bubble growth and hence q_{t+1} (which follows from the fact that the bubble is a state variable). But also, the sum of the fundamental-price and bubble-pledgeability externalities in the bubbly economy is different than the fundamental-price externality in the alternative bubbleless economy with q'_{t+1}. Using (35) we can see that

$$\mathcal{F}_{t+1}q_{t+1} + \mathcal{W}_{t+1}\frac{dB_{t+1}}{dl_{t+1}} = \mathcal{F}_{t+1}\left(q_{t+1} + B_{t+1}\cdot\mathcal{Y}_{t+1}\right) \neq \mathcal{F}_{t+1}\left(q_{t+1} + B_{t+1}\right) = \mathcal{F}_{t+1}q'_{t+1}$$

because $\mathcal{Y}_{t+1} = q_{t+1}\mathcal{W}'_{t+1}/B_{t+1}dB_{t+1}/dl_{t+1} \neq 1$.

[23] For example, Caballero and Krishnamurthy (2006) show that the burst of a bubble used as a store of value can induce a reversal in capital flows and drop in production in emerging economies, while Kocherlakota (2009) studies the distributional and aggregate inefficiencies of bursting bubbles. More recently, Miao and Wang (2015) study how a collapse of a (stochastic) bubble in bank valuations can generate a financial crisis, while Biswas, Hanson and Phan (2020) show how the collapse of a bubble coupled with wage rigidities can lead to a persistent recession.

[24] Clearly, such a situation would also require that the two economies have different fundamentals otherwise asset valuations and, hence, the macroprudential taxes would be different for the same level of debt across the two economies.

5 Quantitative analysis

This section presents the quantitative implications of the model. We proceed by discussing the baseline calibration, and then turn to the numerical results.

5.1 Calibration

We calibrate most of the the parameters following Bianchi-Mendoza to allow for close comparisons of our results with their bubbleless economy. Thus, for calibration details of all non-bubble parameters we refer the reader to their section III.A. One exception is the global interest rate R, which, in order to limit the number of states for the numerical solution, we keep fixed at its long-term average level, $\bar{R} = 1.01$.

The parameters associated with the bubble are calibrated following the results in Jordà, Schularick and Taylor (2015), who study bubbles in equities and housing markets in 17 advanced countries over the past 140 years. They report summary statistics separately for house and for equity bubbles, in pre- and post-World War II periods. Given the sample period used for calibrating other model parameters, we focus on the post-World War II equity bubbles. The average duration of an equity bubble in this period is 2 years, which implies a survival probability of $\pi = 0.5$ in our model. For a period of price growth to be identified as a bubble, Jordà, Schularick and Taylor (2015) require that the log of real asset prices diverge by more than one standard deviation from a country-specific Hodrick-Prescott filtered trend ($\lambda = 100$, annual data). This implies an average deviation of equity prices from their long-term trend of around 9.2 percent in the first year of the bubble. Thus, while we solve the model for a grid of different bubble states, when presenting the results we focus on the state in which the equity bubble is equal to 9.2 percent of the average fundamental price. The latter is computed from the ergodic distribution in the bubbleless economy.

Table 1 summarizes all parameter values. The functional forms for preferences and technology are

$$U = \frac{\left(c - \chi \frac{l^{1+\omega}}{1+\omega}\right)^{1-\sigma} - 1}{1-\sigma}, \quad \omega, \chi > 0, \sigma > 1,$$
$$F = e^z k^{\alpha_k} v^{\alpha_v} l^{\alpha_l}, \quad \alpha_k, \alpha_v, \alpha_l > 0, \quad \alpha_k + \alpha_v + \alpha_l \leq 1.$$

Total factor productivity (TFP) follows an independent AR(1) process, given by

$$z_t = \bar{z} + \rho_z z_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma_\epsilon)$$

The productivity shock is discretized using the Tauchen's quadrature method with three realizations $z^l, \bar{z}, z^h$, such that $z^l < \bar{z} < z^h$. The parameter in the borrowing constraint, m_t, follows a two-state regime switching Markov process with two states $\{m^l, m^h\}$, where m^l denotes tight and m^h denotes normal credit conditions, respectively. This process is also assumed to be independent from the Markov process for z. Finally, $P_{x,y}$ in Table 1 denotes the transition probability from a state x to a state y.

Table (1) Calibration.

Parameters	Value
Risk aversion	$\sigma = 1$
Share of intermediate good in output	$\alpha_v = 0.45$
Share of labor in output	$\alpha_l = 0.352$
Share of assets in output	$\alpha_k = 0.008$
Labor disutility coefficient	$\chi = 0.352$
Fischer elasticity	$1/\omega = 2$
Working capital coefficient	$\theta = 0.16$
Tight credit regime	$m^l = 0.75$
Normal credit regime	$m^h = 0.90$
Global interest rate	$R = 1.01$
TFP process	$\rho_z = 0.78$
	$\sigma_\epsilon = 0.01$
Discount factor	$\beta = 0.95$
Transition probability, m^h to m^l	$P_{h,l} = 0.1$
Transition probability, m^l to m^l	$P_{l,l} = 0$
Bubble bursting probability	$\pi = 0.5$

5.2 Numerical results

To solve the model, we use a global, non-linear solution algorithm. The CE solution is obtained by iterating over the first-order conditions, and the SP problem solution is obtained by applying a value function iteration algorithm.[25] As mentioned at the end of section, we augment the otherwise standard global solution algorithm to incorporate a non-stationary stochastic bubble. We refer the reader to Appendix A.7 for details on the numerical method.

In both CE and SP algorithms, we use a grid of 150 points for L_t over a range $[0, 0.11]$, with 60 states (3 productivity states, 2 credit condition states, and 10 bubble states: one bubbleless state and 9 states with different bubble size of up to over 20 percent of the fundamental price).

Policy functions and optimal macroprudential tax. We now move to the numerical analysis and discuss the optimal policy rules. Figure 2 shows new borrowing L_{t+1} and the total collateral value of the firm, which combines the fundamental price of capital q_t and the bubble component B_t, as functions of the outstanding debt level, L_t, when financial conditions are favorable and productivity is low ($m_t = m^h$ and $z_t = z^l$). The top panels show the bubbleless state, and the bottom panels show the state with a bubble present. CE (SP) decision rules are depicted in red (blue). The vertical red (blue) lines mark outstanding debt levels above which the borrowing constraint starts to bind at t in the CE (SP).

There are notable differences between the CE and SP decision rules. The SP chooses new borrowing L_{t+1} that is always lower than in the CE, independently of the bubble's presence. It does so because it internalizes pecuniary externalities, and mitigates their negative impact on consumption, asset prices and welfare. Moreover, the borrowing constraint starts binding in the SP equilibrium for lower levels of debt than in the CE. This happens as lower SP bor-

[25] In order to obtain the solution for the competitive economy, we iterate the (competitive) Euler equation for borrowing, which does not incorporate the externality terms. On the contrary, the effect of these externalities on welfare is included under value function iteration that yields the planner's solution. Given that we solve for time-consistent policies, we use a nested fixed point algorithm for the value function iteration.

rowing results in lower consumption, putting a downward pressure on collateral values, thereby tightening the borrowing constraint further.

With a bubble, the collateral value of the firm is higher than in the no-bubble case for both SP and CE. As a higher collateral value relaxes the borrowing constraint, a higher level of outstanding debt can be supported. In particular, the borrowing constraint starts binding at t for a threshold of debt that is higher for both CE and SP in the economy with a bubble compared to the one without (the vertical lines in Figure 2 move further to the right). The outstanding debt, L_t, can increase by about 40 percent more compared to the bubbleless case before the borrowing constraint starts binding at t. This result is crucial for how the planner chooses to mitigate debt growth. As discussed earlier, reducing new borrowing alleviates pressures on future valuations, but intensifies pressures on valuations in the current period. The bubble amplifies the importance of this trade-off.

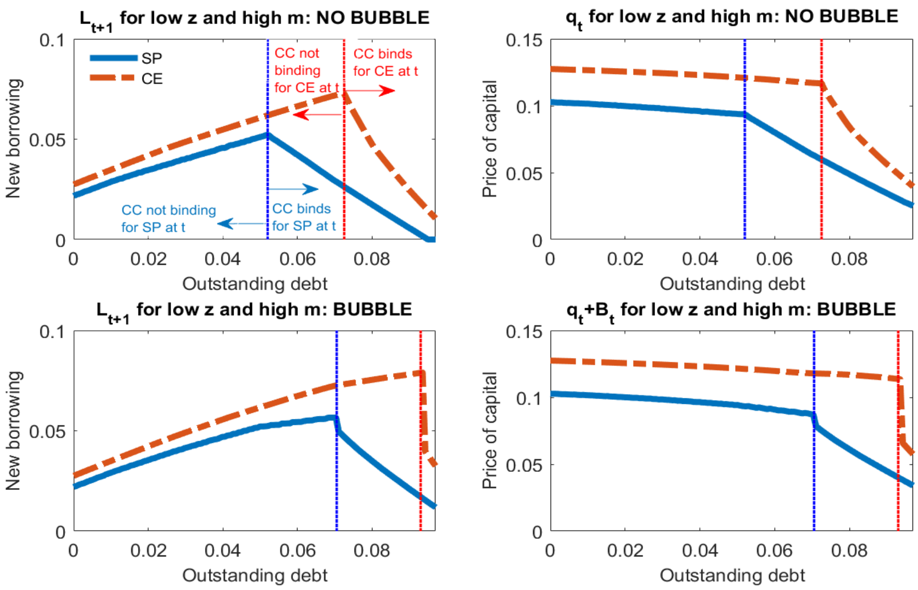

Figure (2) Policy rules for borrowing and collateral values in the presence and absence of a bubble.

Note: The figure plots the new borrowing L_{t+1} (panels on the left) and collateral values (panels on the right) as a function of outstanding debt L_t, for $z_t = z^l$ and $m_t = m^h$. The top panels show the case without a bubble, and the bottom panels show the case with a bubble of 9.2% of the average fundamental price in the bubbleless economy. The blue lines correspond to the SP policy rules, while the red dashed lines represent the CE policy rules. The borrowing constraint binds for outstanding debt levels to the right from the red (CE) and blue (SP) vertical lines.

The left panel in Figure 3 plots the optimal tax on borrowing when there is a bubble (solid line) and when there is no bubble (dotted line), as a function of outstanding debt, L_t. The tax corresponds to low productivity and normal credit conditions, and a bubble state such that the bubble is equal to 9.2% of the average fundamental price in the bubbleless economy. The right panel in Figure 3 shows the contribution of the current and future extensive and intensive margins to the difference of the macroprudential tax in the bubbly economy and the borrowing tax in the bubbleless economy given by equation (40). A couple of observations are worth noting.

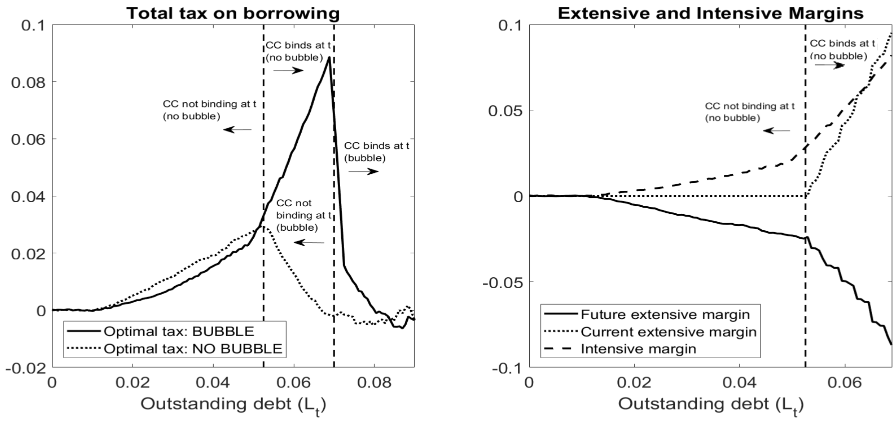

Figure (3) Optimal tax on borrowing in the presence of a bubble.

Note: The left panel figure plots the optimal tax at t in the absence of a bubble and in the presence of a bubble, b_t, equal to 9.2% of the fundamental price as a function of debt outstanding, L_t, when z_t is low and m_t is high. The right panel figure plots the contribution of the extensive and intensive margins in the difference between the macroprudential tax in the presence of a bubble and the borrowing tax absent a bubble as a function of debt outstanding, L_t, when z_t is low and m_t is high. We report the margins for the region of L_t where the tax in the presence of a bubble takes the macroprudential interpretation, i.e. the borrowing constraint does not bind at t in the bubbly economy.

First, for very low levels of debt the tax on borrowing is zero, independently of whether there is a bubble or not. The reason is that for such low levels of outstanding debt the borrowing constraint does not bind at t or $t+1$, irrespective of which state of the world realizes. In other words, there is no need for a policy intervention when credit imbalances are subdued.

Second, for low to medium levels of outstanding debt, the optimal tax is lower when the bubble is present. In this region, the borrowing constraint is slack at t for both the bubbly and bubbleless economies. As can be seen from the figure on the right, the future extensive margin, which is negative, dominates the intensive margin, which is positive, and pushes the tax down in the bubbly economy. In other words, for low to moderate levels of outstanding debt, the benefits of the bubble from reducing the probability of binding borrowing constraints in the future outweigh the costs accruing from the bubble-pledgeability and bubble-accumulation externalities when future constraints bind. This justifies accommodating the bubble by levying a relatively lower tax compared to the one in the bubbleless economy.

Third, as Figure 3 shows, for L_t of around 0.052, the borrowing constraint becomes binding at t in the absence of the bubble. As a result, the SP reduces the tax in the bubbleless economy in order to weigh the benefit of relaxing the constraint today against the cost of it binding in the future. In other words, besides the forward-looking macroprudential component, the tax now also incorporates a contemporary component, that targets the binding borrowing constraint today. The tax becomes smaller for higher levels of outstanding debt and can turn slightly negative in our calibration. On the contrary, the optimal tax continues to increase in the presence of the bubble until the level of outstanding debt reaches levels of around 0.07. The tax continues to be macroprudential in nature, due to the bubble's positive effect on relaxing the borrowing constraint at t and it rises to much higher levels than in the bubbleless economy (a maximum of about 8.8 percent compared to about 3 percent when there is no bubble).

The contribution of the *current* extensive margin to the difference between the taxes in the bubbly and bubbleless economies is positive. Similarly, the intensive margin contributes positively to this difference as the future bubble-pledgeability and bubble-accumulation externalities become more severe for higher L_t. On the contrary, the *future* extensive margin contributes negatively to the difference between the two tax rates, owing mainly to higher future (fundamental-price) externalities in the bubbleless economy for higher L_t; yet, it is not strong enough to dominate the combined current extensive and intensive margins, which push the tax differential in the opposite direction. Hence, asset overvaluations amplify credit imbalances when debt is high, calling for a (much) higher tax in the bubbly economy.

An alternative way to analyze the difference in the tax rates between the two economies is to consider the net effect of the current and future extensive margins as reported in the left chart in Figure 4. The net extensive margin starts to increase—while it continues being negative—at the point when the constraint binds in the bubbleless economy, indicated by the vertical dashed line. Yet, the tax differential between the bubbly and bubbleless economies is positive as already shown in Figure 3. This suggests that the macroprudential tax in the presence of the bubble is largely driven by the intensive margin beyond the level of indebtedness for which the constraint binds in the bubbleless economy (right chart in Figure 4). For higher levels of indebtedness, the net extensive margin eventually becomes positive, contributing further to the higher macroprudential tax in the presence of the bubble.

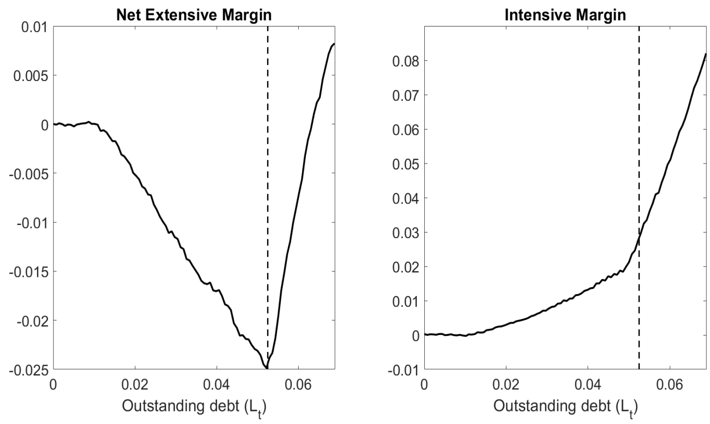

Figure (4) Net Extensive and Intensive Margins.

Note: The left chart plots the sum of the current and future extensive margins as a function of debt outstanding, L_t, when z_t is low and m_t is high. The right chart plots the intensive margin. We report the margins for the region of L_t where the tax in the presence of a bubble takes the macroprudential interpretation, i.e. the borrowing constraint does not bind at t. The vertical dashed line indicates the level of L_t for which the constraint starts binding in the bubbleless economy.

Figure 5 shows the credit-asset valuations feedback loop from another angle. It plots the normalized optimal tax as a function of the bubble size for three levels of outstanding debt L_t.[26] A first observation is that for the same exogenous states z_t and m_t, and endogenous state L_t, the level of the borrowing tax differs across the various bubble states b_t in all three cases reported. Hence, macroprudential regulation should not only take into considerations the fundamentals of the economy, z_t and m_t, and the level of debt, L_t, but also the bubbly valuations. Second,

[26]We have normalized the tax corresponding to the bubbleless case for each L_t to 100 in order to easily show how the bubble size pushes the tax up or down (naturally the no-bubble tax is different for different levels of L_t).

whether the bubble pushes the tax up or down depends on the level of the outstanding debt. As already discussed in detail, there are both positive and negative aspects associated with a bubble. Consistent with our earlier analysis, the positive aspects outweigh the negative ones for lower levels of debt and the tax is decreasing in the size of the bubble. For higher levels of debt the negative effects of the bubble become stronger and at some point they dominate, resulting in an optimal tax that is increasing in size of the bubble.

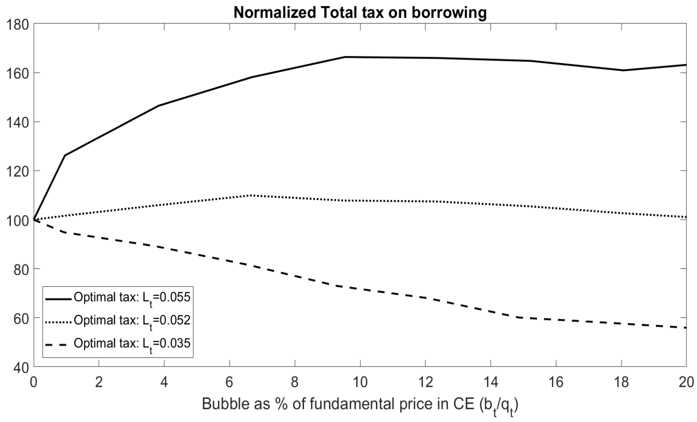

Figure (5) Optimal tax on borrowing in the presence of an asset price bubble.

Note: The figure plots the normalized optimal tax at t as a function of the bubble size for three levels of debt outstanding: $L_t = 0.035$ (dashed line), $L_t = 0.052$ (dotted line), and $L_t = 0.055$ (solid line) when z_t is low and m_t is high. We have chosen to normalize the tax in the absence of a bubble to 100 for each of the three levels of debt considered.

In sum, our results suggest that asset price overvaluations matter for optimal regulation, but might not be enough by themselves to justify a tightening of macroprudential policy. This happens because for low or moderate levels of credit, the bubble's persistence into the next period makes tight financial conditions less likely in the future, while the costs of the bubble deflating in the future are still low. As a result, under our calibration parameters, it is optimal not to lean against the bubble when the level of debt is low or moderate. However, once debt increases sufficiently, the macroprudential policy should be stronger.

Optimal tax and probability of bubble bursting. In this subsection, we investigate how the probability that the bubble bursts, $1 - \pi$, affects the level of optimal policy. The left panel in Figure 6 shows the macroprudential tax for different levels of π (probability of a bubble continuing) as a function of the outstanding level of debt, L_t, when the value of the bubble today, b_t, is equal 9.2 percent of the fundamental price (our benchmark case above).

The results suggest that the optimal macroprudential tax is higher when the probability of a bubble bursting is higher (or, equivalently, π is smaller) for all levels of L_t. The reasons why a higher probability of bursting calls for a higher macroprudential tax are intuitive. First, the borrowing constraint is more likely to bind in future states when π is lower, which pushes the future extensive margin component of the tax up (i.e. makes the contribution of the extensive margin less negative), as shown in the middle panel in Figure 6. The intensive margin component of the tax also increases as shown in the right panel in Figure 6. The reason is that, for a lower π, a bigger bubble is required in the future to support the same bubble valuation today. In turn, a larger value of the bubble in the future translates into more severe bubble externalities when the constraint binds and, hence, a higher macroprudential tax today to tackle them. Overall, the optimal macroprudential policy is more aggressive for "riskier" stochastic bubbles, as measured by the probability of bursting.[27]

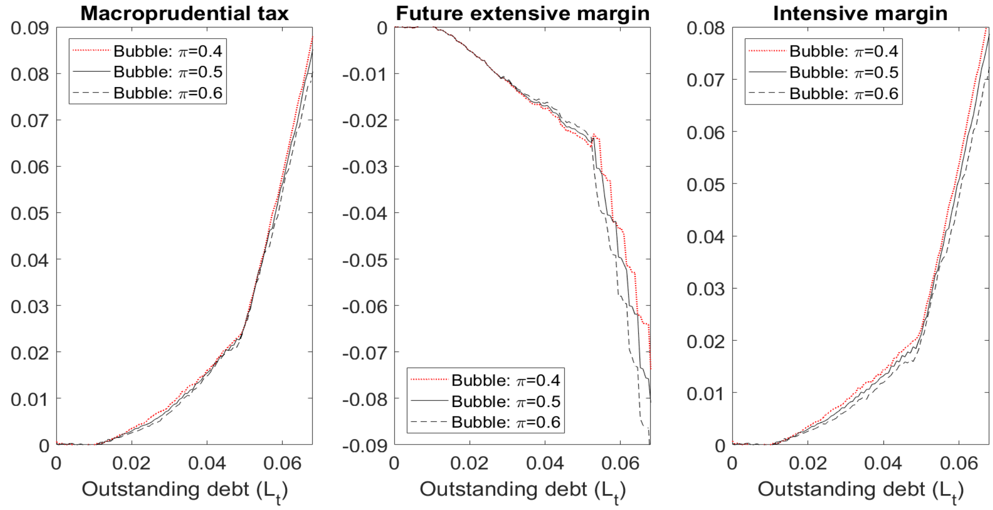

Figure (6) Macroprudential tax on borrowing and probability of bubble bursting.

Note: The figure plots the macroprudential tax for $\pi = 0.4$, $\pi = 0.5$, and $\pi = 0.6$, as well as the contribution of the future extensive and intensive margins as a function of debt outstanding, L_t when z_t is low and m_t is high.

Crisis scenario simulations. Next, we perform a simulation exercise to investigate the magnitude of the externalities from bubble.

The competitive economy is simulated for 11 periods, but to reduce dependence on the initial conditions, we drop the initial period when computing average responses. In the first period, a bubble with a size of around 4% deviation relative to the fundamental price of capital is present and the bubble's continuation probability is set at $\pi = 0.5$.[28] The 11-period path is then simulated 100,000 times. We do not simulate the economy once for a very high number of periods because the bubble in our setting does not re-emerge after bursting, generating an absorbing state. Thus, the distribution of outcomes would be biased towards the no-bubble outcomes.

We are interested in the economy's responses to a binding borrowing constraint in three cases: (i) there is no bubble before, during or after the borrowing constraint binds, (ii) there is a bubble that persists throughout the period when the borrowing constraint binds, and (iii) the bubble pops when the borrowing constraint starts to bind. Figure 7 shows average responses of consumption, new borrowing, and the collateral value of the firm—all in terms of deviations from averages across all simulations—for these three events.

[27]The current extensive margin is the same across all bubbly economies as it does not depend on π, but only on the bubbleless equilibrium allocations.

[28]We set the initial bubble size at 4% as it corresponds to an expectation of a bubble of around 9% of fundamental price in the next period

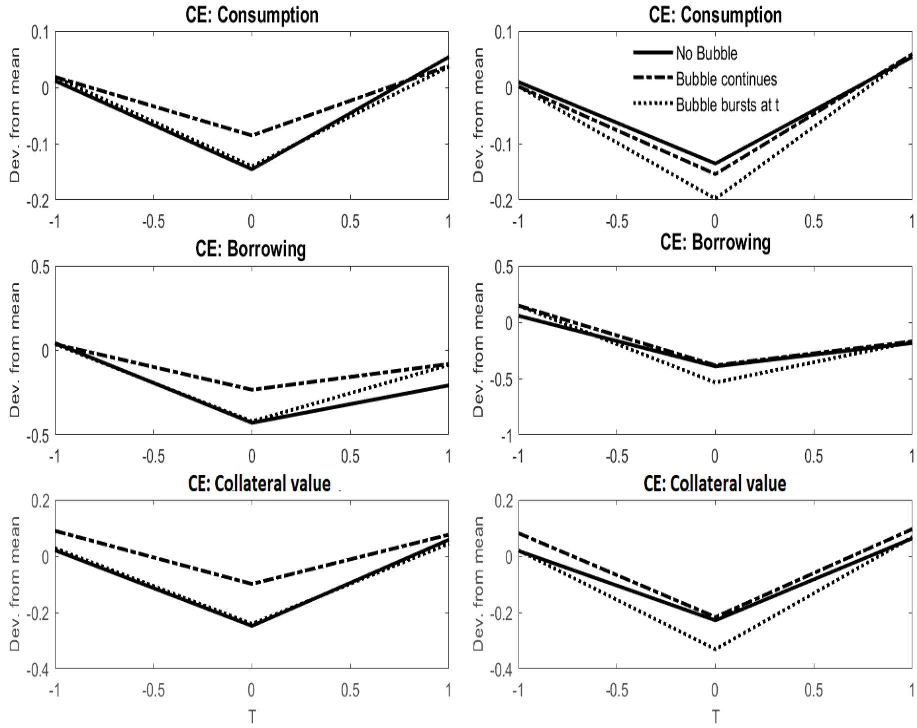

Figure (7) Model simulations: Responses of key variables to a binding credit constraint.

Note: The figure plots responses of consumption c_t, new borrowing L_{t+1}, and collateral value of the firm $q_t + B_t$ in the competitive economy (CE), in the event of the borrowing constraint binding ($T = 0$ in the figure), in presence of a bubble that persists throughout the event (dashed line), in the case when the bubble pops in the same period when the borrowing constraint binds (dotted lines), and in the absence of a bubble (solid line). The left panels show responses when the starting level of debt outstanding in the 11-period simulated path is low ($L_1 = 0.045$), the right panels—when the initial debt level is high ($L_1 = 0.095$). All responses are in terms of deviation from averages across all simulations.

As we can simulate the economy repeatedly only for a limited number of periods, the level of outstanding debt in the first period is likely to matter for the net effect of the bubble and for our event comparisons. Thus, the left-hand side of Figure 7 shows consumption, borrowing, and the collateral value when the period-one outstanding debt is low, and the right-hand side when it is high.

As expected, with a relatively subdued starting level of debt, the bubble has a positive effect on consumption and collateral values when the borrowing constraint becomes binding. In contrast, when the initial level of debt is high, the bubble on average amplifies the negative effects of a binding constraint as consumption and borrowing fall somewhat more when the bubble persists compared to the no-bubble case. Finally, all variables experience the biggest declines when the bubble pops and the borrowing constraint becomes binding at the same time. Intuitively, the existence of the bubble in the preceding periods allows firms to build more debt making the effect of a binding constraint more severe.

Welfare comparisons. Next, we analyze the social planner's allocations and the welfare effects of the planner's tax on borrowing. The non-recurrent character of the bubble and the small number of simulated periods are limitations for welfare comparisons. In order to mitigate the dependence of the welfare analysis on the initial conditions, we set the initial value of the outstanding debt level equal to its long-term average, which is generated from the ergodic distribution of debt in a bubbleless economy and is equal to around $\bar{L} = 0.07$.[29]

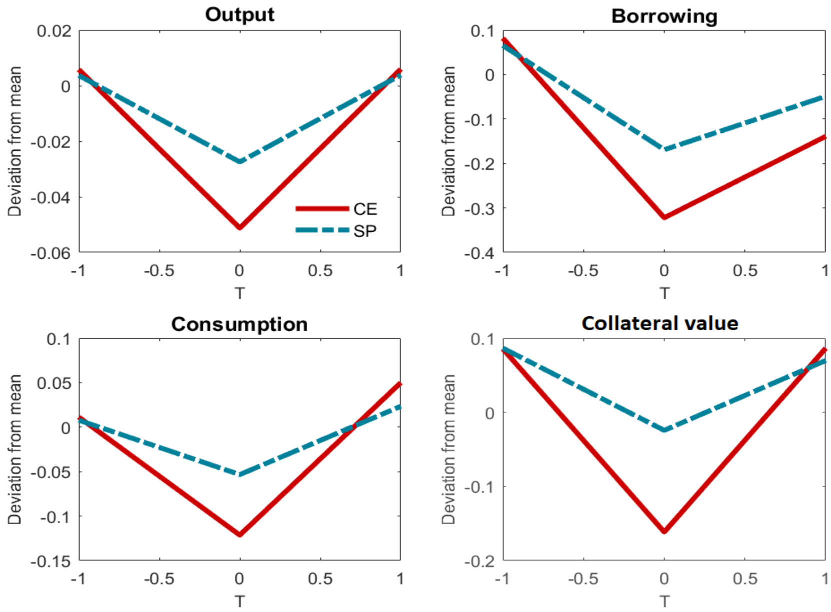

Figure (8) Model simulations: Social Planner's allocations.

Note: The figure plots responses of output y_t, new borrowing L_{t+1}, consumption c_t, and capital price q_t in competitive economy (CE, red solid line) and under a social planner (SP, blue dashed line), across states in which the borrowing constraint binds ($T = 0$ in the figure), and when there is a bubble in competitive economy. The debt level in the initial period is set to 0.07. All responses are in terms of deviation from averages across all simulations, but excluding the initial period.

Figure 8 shows the average responses of output, borrowing, consumption, and the collateral value of the firm in the competitive economy and for the social planner. We focus on the states when the borrowing constraint binds and the bubble persists.[30] It is easy to see that, as the social planner prevents excessive borrowing ex-ante, it is able to considerably mitigate the impact of the binding constraint on debt deleveraging, consumption and output as well as on collateral values.

Moreover, we compute the welfare gain in the planner's economy as the average compensating consumption variation that equalizes the expected utility between the CE and SP. Table 2 summarizes the results.

The top panel of Table 2 shows the summary statistics for the three events depicted in Figure 7 when the starting level of debt is equal to the long-term average. As before, the drop in collateral values is the largest when the bubble bursts and the borrowing constraint binds at the same time.

The average level of the tax on borrowing is around 1.7 percent, and it reaches a maximum of 5.3 percent (bottom panel of Table 2). The welfare gain from the optimal tax policy is around

[29]The ergodic distribution is obtained from a simulation of the bubbleless competitive economy for 100,000 periods.

[30]To do that we repeat the short-period simulations, for the same paths of realized states of the exogenous variables, using CE and SP policy functions.

Table (2) Summary statistics.

Competitive Economy: Event Statistics	
CC binds and there is no bubble	
Probability	7.4%
Collateral value drop	-28.0%
CC binds and bubble persists	
Probability	0.4%
Collateral value drop	-22.9%
CC binds and bubble pops	
Probability	0.7%
Collateral value drop	-34.9%
Social Planner: Welfare Effects	
Minimum tax on debt	0.0%
Average tax on debt	1.7%
Maximum tax on debt	5.3%
Welfare gains	0.4%

Note: The table shows summary statistics of 100,000 simulations of 11-period paths when the initial period level of debt is set at 0.07. CC stands for borrowing constraint.

0.4 percent. While this number might seem small, it is important to note that the welfare effects of optimal policies in representative-agent models with the CRRA preferences are, generally, small.

Finally, we use our simulations to examine whether the bubble improves average welfare. To do so, we re-run our simulations for a bubbleless economy for the same paths of realized states of exogenous variables and an initial debt level of 0.07. Comparing the new CE allocations with our previous simulations, we find that the net welfare effect of the bubble is negative, at around -0.1 percent. That is, the negative effects from the bubble outweigh the positive effect from relaxing the borrowing constraints, at least in this experiment.

Simple rules. The optimal policy is implemented by a state-contingent tax, which may raise some concerns about practical implementation. Thus, we examine whether policies based on simple rules can generate welfare gains too. In principle, simple rules could be beneficial if they share some characteristics of the optimal policy, such as leaning against credit imbalances and asset overvaluations. We consider three types of simple rules. First, a time-invariant tax, which takes the form $\bar{\tau}$, and it equal to the average (state-contingent) tax in the planner's simulated economy (equal to 1.7 percent as reported in Table 2). Second, a tax that depends only on deviations of debt from the average of the ergodic distribution of debt in the bubbleless economy ($\bar{L}$), and takes the from $\tau_t = (1+\tau_0)\left[(L_t/\bar{L})^{\eta_1}\right]-1$; third, a tax that depends both on deviations of debt and valuations from their ergodic average in the bubbleless economy ($\bar{q}$), and takes the form $\tau_t = (1+\tau_0)\left\{(L_t/\bar{L})^{\eta_1}\cdot[(q_t+B_t)/\bar{q}]^{\eta_2}\right\}-1$.

Table 3 reports the welfare gains under these three simple rules for combinations of $\eta_1 \in \{0, 0.5, 1\}$ and $\eta_2 \in \{0, 0.5, 1\}$. There are three key takeaways. First, a small invariant tax, corresponding to $\eta_1 = \eta_2 = 0$, can increase welfare in our simulated economy.[31] Second, incorporating credit or valuation imbalances can improve upon the invariant tax, which correspond

[31] This is not a general result and may depend on the specifics of our simulation. In Bianchi and Mendoza (2018) the invariant tax is inefficient albeit their economy does not feature a bubble.

to simple rules with $\eta_1 > 0$ and $\eta_2 = 0$, or $\eta_1 = 0$ and $\eta_2 > 0$, respectively. Third, incorporating both credit and valuation imbalances further increases the welfare gains, which get closet to (but still lower than) the gains under the optimal policy equal to 0.4 percent (as reported in Table 2); this is also true for simple rules with $\eta_1, \eta_2 > 1$ (not shown).

Table (3) Simple rules' welfare gains.

		η_2		
		0.0	*0.5*	*1.0*
	0.0	0.09%	0.19%	0.22%
η_1	*0.5*	0.13%	0.22%	0.24%
	1.0	0.14%	0.18%	0.23%

Note: The table shows welfare gains in terms of compensating consumption variations.

6 Conclusions

We study optimal macroprudential policy when credit imbalances associated with firms' borrowing are accompanied by stock market bubbles. We show that the presence of a bubble generates additional externalities, which, *ceteris paribus*, requires a more aggressive macroprudential intervention in order to avoid a bubble deflation in bad times, that is when the bubble is most useful to relax borrowing constraints. But, at the same time, the presence of the bubble alters equilibrium allocations, and by helping to keep collateral constrains slack for some shock realizations, it may result in a macroprudential tax on borrowing that is lower relative to the bubbleless case. Our quantitative results suggest that the optimal policy response depends, in a non-monotone way, on the outstanding level of debt in the economy. When credit imbalances are moderate, the optimal tax is lower in the presence of the bubble: While the bubble relaxes the borrowing constraint, externalities associated with the bubble are not so severe as agents do not need to deleverage much, if the constraint binds in the future. However, when the credit imbalances are high, the optimal tax level is much higher than in the absence of the bubble, in order to tackle the amplifying effect on future deleveraging from elevated credit imbalances and asset overvaluations.

The above results have important implications for the determination of countercyclical policies targeting credit imbalances employed by regulators globally. In particular, asset overvaluations should not always be an argument to lean more aggressively against the wind, but could also imply that regulators need not to worry as much about the build-up of price overvaluations if the existing credit imbalances are not high.

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A Appendix

A.1 Derivation of borrowing constraint

The collateral constraint (9) can be derived from a debt renegotiation problem between a firm and its lenders. After the outstanding debt from last period, L_t, has been repaid, but before production and investment in new capital take place, the firm can try to renegotiate the outstanding debt, equal to $L_{t+1}/R + \theta p^v v_t$ at that point. Following Jermann and Quadrini (2012), we assume that the firm has full bargaining power. If lenders do not agree to renegotiate the debt, the firm diverts the borrowed funds in full. Lenders can then try to seize the firm, restructure it by canceling the debt, and sell it back to households in the equity market within period t. The restructured firm does not pay any dividends at t, thus households will value it only according to its future value at $t + 1$. In turn, the market value of the restructured firm with (existing) capital k_t and zero debt is given by $V_{t+1}(k_t, 0, b_{t+1})$ in each state at $t + 1$. Thus, the price that households would be willing to pay for the restructured firm at t is equal to its expected market price at $t + 1$ evaluated at their stochastic discount factor, i.e. $E_t[\beta U_{c,t+1}/U_{c.t} V_{t+1}(k_t, 0, b_{t+1})]$. Lenders can only successfully seize the firm with probability m_t, which can be interpreted as the probability that the lenders can indeed monitor and enforce the relevant covenants. It follows that the expected value of the seized firm from the lenders' perspective is given by $m_t E_t\left[\beta U_{c,t+1}/U_{c,t} V_{t+1}(k_t, 0, b_{t+1})\right]$, which is their outside option during bargaining.

We first examine the expected surplus from renegotiation, V_t^R. Firms have all bargaining power and can, thus, extract all surplus while renegotiating the debt at t apart from the value of the lenders' outside option. If lenders receive a payment during the renegotiation process equal to their outside option, then the seizure and restructuring of the firm can be avoided. Hence, for the firm managers, who maximize the firm's value, V_t^R is equal to the total amount borrowed minus the transfer they need to make to lenders so that the latter do not exercise their outside option plus the expected firm's value given that restructuring has been avoided

$$V_t^R = \overbrace{L_{t+1}/R + \theta p^v v_t}^{\substack{\text{Borrowed funds firm} \\ \text{tries to renegotiate}}} \overbrace{-m_t E_t\left[\beta \frac{U_{c,t+1}}{U_{ct}} V_{t+1}(k_t, 0, b_{t+1})\right]}^{\substack{\text{Value of lenders'} \\ \text{outside option}}} + \overbrace{E_t\left[\beta \frac{U_{c,t+1}}{U_{ct}} V_{t+1}(k_{t+1}, L_{t+1}, b_{t+1})\right]}^{\substack{\text{Expected firm's value if} \\ \text{seizure and restructuring are avoided}}}. \tag{A.1}$$

On the contrary, the expected surplus if firm managers decide not to renegotiate the debt, V_t^{NR}, is just equal to the firm's expected value:

$$V_t^{NR} = \overbrace{E_t\left[\beta \frac{U_{c,t+1}}{U_{c,t}} V_{t+1}(k_{t+1}, L_{t+1}, b_{t+1})\right]}^{\substack{\text{Expected firm's value} \\ \text{absent renegotiation}}}. \tag{A.2}$$

The incentive compatibility constraint requires that the expected surplus after renegotiation is smaller or equal to the expected firm's value if renegotiation does not occur, i.e. $V_t^{NR} \geq V_t^R$. Because the expected value of the firm absent restructuring given renegotiation and absent renegotiation is the same, this incentive compatibility constraint gives rise to the following borrowing constraint

$$L_{t+1}/R + \theta p^v v_t \leq m_t E_t\left[\beta \frac{U_{c,t+1}}{U_{ct}} V_{t+1}(k_t, 0, b_{t+1})\right], \tag{A.3}$$

which we have reported in (9).

The assumption that lenders sell the restructured firm back to households within period t is important. If, instead, we had assumed that the restructured firm is sold back to the market at $t+1$, then the stochastic discount factor of lenders, $1/R$, should have been used to obtain the period t value to lenders of the restructured firm equal to $E_t[V_{t+1}(k_t, 0, b_{t+1})/R]$. Thus, the borrowing constraint in that case should have be written as $L_{t+1}/R+\theta p^v v_t \leq m_t E_t[V_{t+1}(k_t, 0, b_{t+1})/R]$ (see Martin and Ventura, 2018, for a macroeconomic model of bubbles with such a constraint). Although considering this alternative borrowing constraint should not matter qualitatively for our results, it would not replicate the Bianchi-Mendoza economy absent a bubble and would complicate the solution for the firm's value function. Given that we want our bubbleless economy to be equivalent to the BM economy to facilitate a clean comparison, we assume that lenders do not hold the restructured firm intertemporally, but sell it back in the equity market in the same period. In particular, we assume that it is costly for lenders to hold the restructured firm across periods, i.e. their outside option from selling the firm within period is more valuable, giving rise to (A.3).

A.2 Conjecture and Verify: Firm's Value Function

Conjecture that the representative firm's value function is given by $V_t(k_t, L_t, b_t) = a_t k_t + s_t L_t + b_t$, where a_t, s_t and b_t are time-varying coefficients to be determined. Substituting into the firm's optimization problem (7) and into constraints (8)-(9) yields the following optimization problem

$$a_t k_t + s_t L_t + b_t = \max_{k_{t+1}, L_{t+1}, l_t, v_t} D_t + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} (a_{t+1} k_{t+1} + s_{t+1} L_{t+1} + b_{t+1}) \right], \tag{A.4}$$

subject to

$$D_t = z_t F(k_t, l_t^d, v_t) - p^v v_t - w_t l_t^d + \frac{L_{t+1}}{R} - L_t + q_t k_t - q_t k_{t+1}, \tag{A.5}$$

and

$$\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} (a_{t+1} k_t + b_{t+1}) \right]. \tag{A.6}$$

Taking first-order conditions with respect to $k_{t+1}, L_{t+1}, l_t^d, v_t$ respectively, yields

$$q_t = \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} a_{t+1} \right], \tag{A.7}$$

$$\frac{1}{R} = \frac{\mu_t}{R} - \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} s_{t+1} \right], \tag{A.8}$$

$$w_t = z_t F_{l,t}, \tag{A.9}$$

$$p^v = z_t F_{v,t} \frac{1}{1+\theta \mu_t}. \tag{A.10}$$

Substituting these conditions back into (A.4) and simplifying, yields

$$\begin{aligned} a_t k_t + s_t L_t + b_t = {} & z_t F(k_t, l_t^d, v_t) - z_t F_{l,t} l_t^d - z_t F_{v,t} v_t - L_t + q_t k_t \\ & + \mu_t m_t \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} (a_{t+1} k_t + b_{t+1}) \right] + \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1} \right], \end{aligned} \tag{A.11}$$

where we have used the complementary slackness condition

$$\mu_t \left(\frac{L_{t+1}}{R} + \theta p^v v_t - m_t \beta E_t \left[\frac{U_{c,t+1}}{U_{c,t}} (a_{t+1} k_t + b_{t+1}) \right] \right) = 0 \tag{A.12}$$

to substitute out the $\mu_t(L_{t+1}/R + \theta p^v v_t)$. Comparing both sides of equation (A.11) yields

$$a_t = z_t F_{k,t} + q_t(1 + m_t\mu_t), \tag{A.13}$$

$$s_t = -1, \tag{A.14}$$

$$b_t = (1 + m_t\mu_t)\beta E_t\left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1}\right], \tag{A.15}$$

where $F_{k,t} = F(k_t, l_t^d, v_t) - F_{l,t}l_t^d - F_{v,t}v_t$. Thus, the value function takes the form

$$V_t(k_t, L_t, b_t) = \left(z_t F_{k,t} + q_t(1 + m_t\mu_t)\right) k_t - L_t + b_t. \tag{A.16}$$

Moreover, using the above value function to substitute out a_{t+1}, yields the credit constraint

$$\frac{L_{t+1}}{R} + \theta p^v v_t \leq m_t(q_t k_t + B_t). \tag{A.17}$$

A.3 Proof of Proposition 1

Using that $a_{t+1} = z_{t+1}F_{k,t+1} + q_{t+1}(1 + m_{t+1}\mu_{t+1})$ and $s_{t+1} = -1$, the first-order conditions (A.7)-(A.8) can be rewritten as

$$U_{c,t}q_t = \beta E_t\left[U_{c,t+1}\left(z_{t+1}F_{k,t+1} + q_{t+1}(1 + m_{t+1}\mu_{t+1})\right)\right], \tag{A.18}$$

$$\mu_t = 1 - \beta R E_t\left[\frac{U_{c,t+1}}{U_{c,t}}\right], \tag{A.19}$$

yielding, together with equations (A.9)-(A.15), the first-order conditions (18)-(22) in Proposition 1. To derive (23), we combine (A.15) with the definition of B_t in (16).

A.4 Proof of Proposition 2

We, first, show that if $lim_{t\to\infty} \sup \mu_t = 0$, then a rational bubble cannot be supported in equilibrium. Iterating forward the bubble accumulation equation 13, we get

$$\begin{aligned} b_t &= (1 + m_t\mu_t)\beta E_t\left[\frac{U_{c,t+1}}{U_{c,t}} b_{t+1}\right] = \\ &= \lim_{T\to\infty} E_t\left[\beta^T \frac{U_{c,t+T}}{U_{c,t}} b_{t+T} \Pi_{i=0}^{T-1}(1 + m_{t+i}\mu_{t+i})\right]. \end{aligned} \tag{A.20}$$

The transversality conditions implies that

$$\lim_{T\to\infty} E_t\left[\beta^T \frac{U_{c,t+T}}{u_{c,t}} b_{t+T}\right] = 0. \tag{A.21}$$

First, if $\mu_{t'} = 0$ for all $t' > t$, then $b_t = 0$ using (A.20) and (A.21). Second, assume that there is a $T' > t$ such that $\mu_{t'} = 0$ for $t' \geq T'$, but may be positive or zero for $t' \in (t, T')$. Then, $b_t = E_t\left[\beta^{T'} \frac{U_{c,t+T'}}{U_{c,t}} b_{t+T'} \Pi_{i=0}^{T'-1}(1 + m_{t+i}\mu_{t+i})\right] = 0$, because from (A.21) applied to time $t + T'$ we have that

$$b_{t+T'} = \lim_{T\to\infty} E_{t+T'}\left[\beta^T \frac{U_{c,t+T'+T}}{U_{c,t+T'}} b_{t+T'+T}\right] = 0.$$

The first case reflects the situation where the borrowing constraint never binds after time t, while the second case reflects the situation where it occasionally binds for a finite period of time. In both cases, $lim_{t'\to\infty}\sup\mu_{t'}=0$. If, instead, the borrowing constraint binds infinitely often, or $lim_{t'\to\infty}\sup\mu_{t'}>0$ for all t', then $\lim_{T\to\infty}\Pi_{i=0}^{T-1}(1+m_{t'+i}\mu_{t'+i})>1$ in (A.20) for all t' and the transversality condition (A.21) cannot exclude rational bubbles in equilibrium.

Next, we show that there is no $T'\geq t$ such that $\mu_{t'}=0$ for $t'\geq T'$. Our argument is based on comparing the natural debt limit (see Aiyagari, 1993,1994) to the maximum borrowing under the borrowing constraint. The natural debt limit, $\bar{L}$, reflects the level of borrowing that the firm can sustain even if it receives only low productivity realizations and is obtained by iterating forward the firm's budget constraint and setting all dividends D_t to zero (see, also, an early working paper version of Bianchi-Mendoza):

$$\bar{L}\equiv R\frac{[zF_k]_{min}}{R-1}, \tag{A.22}$$

where $[zF_k]_{min}$ is the minimum return on capital. Consider an arbitrarily tighter borrowing limit given by $\tilde{L}\equiv\bar{L}-\delta$, for arbitrarily small δ. Then, the firm can borrow at most $\tilde{L}/R$ at any point in time, which is less than the natural debt limit. Then, using an argument similar to Proposition 3 in Aiyagari (1993), we can show that there are always states such that this debt limit is binding as t goes to infinity. The argument goes as follows: For all levels of debt equal or less than $\tilde{L}$, dividends D_t are strictly positive, as is consumption $c_t=D_t+w_tl_t=D_t+G_{l,t}l_t>0$ using (2), the market clearing condition $\eta_t^j=\eta_{t+1}^j=1$, and the optimality condition (4). As a result, the marginal utility of consumption is equal to $U_{c,t}=U'(c_t-G(l_t))=U'(D_t+G_{l,t}l_t-G(l_t))$ and, thus, it is finite because $D_t+G_{l,t}l_t-G(l_t)>0$.[32] Define the lowest possible level of the composite commodity, $c_t-G(l_t)$, as $\underline{c}=\inf D_t+G_{l,t}l_t-G(l_t)$ and assume that the debt limit never binds. Thus, using the optimality condition with respect to borrowing (18) for $\mu_t=0$, we get $U'(c_t-G(l_t))=\beta RE_t(U'(c_{t+1}-G(l_{t+1})))\leq\beta RU'(\underline{c})<U'(\underline{c})$, because U' is decreasing and $\beta R<1$. If we let $c_t-G(l_t)\to\underline{c}$, this results in a contradiction. In other words, there are always states such that the debt limit introduced by $\tilde{L}$ will bind. Hence, it suffices to show that the our borrowing constraint (15), imposes stricter limits to borrowing than the natural debt limit in order to guarantee that the former will occasionally bind as t goes to infinity.

To establish this, we consider the maximum going concern value of the firm and also take $v_t\to 0$ to make the borrowing constraint as loose as possible. Define by q_{max} and B_{max} the maximum fundamental and bubble collateral values. To obtain the former, assume that only the highest possible productivity realizes deterministically and discount by $1/R$, which is the upper bound for the (non-)stochastic discount factor, since $\beta U_{c,t+i+1}/U_{c,t+i}=(1-\mu_{t+i})/R$, i.e.

$$q_{max}=\frac{[zF_k]_{max}}{R-1}. \tag{A.23}$$

To derive B_{max} suppose that $\mu_{t'}>0$ infinitely often for some $t'>t$. We will show that the borrowing constraint indeed binds, i.e. this assumption does not lead to a contradiction and an equilibrium with a rational bubble does exist.

Define $\overline{B}_t\equiv\beta^tU_{c,t}B_t$. Then, combining (22) and (23), we get that $U_{c,t}B_t=E_t[\beta U_{c,t+1}(1+m_{t+1}\mu_{t+1})B_{t+1}]$ and $\overline{B}_t\geq E_t\overline{B}_{t+1}$. Hence, $\overline{B}_t$ is a super-martingale. Because $\overline{B}_t$ is nonnegative, the super-martingale convergence theorem applies and, thus, $\overline{B}_t$ converges almost surely to a nonnegative random variable, i.e. $\overline{B}_t\to_{a.s.}\overline{B}$. Moreover, $M_t\equiv(\beta R)^tU_{c,t}$ is a super-martingale, because $(1-\mu_t)U_{c,t}=\beta RE_tU_{c,t+1}$, and, thus it converges to a nonnegative random variable. Since $\beta R<1$, $U_{c,t}$ does not need to converge asymptotically but can remain finite and continue to vary randomly (see Ljungqvist and Sargent, 2004, p.574). By the same logic, B_t can remain

[32] It suffices that $G_{l,t}-G(l_t)/l_t>0$. Because $G(0)=0$, $G(l_t)/l_t$ is the slope of the straight line starting at zero and passing through point $(G(l_t),l_t)$. Given that G is strictly convex, its image is always below the image of the straight line for any point $x\in(0,l_t)$. Because G' is strictly increasing, G will necessarily cross the straight line connecting zero and $(G(l_t),l_t)$ from below and, hence, the derivative of G at l_t is strictly higher than the slope of the straight line.

finite and continue to vary randomly, because $\beta < 1$ and $U_{c,t}$ is finite.[33]

Hence, B_{max} is finite and non-zero, and the maximum borrowing that satisfies the borrowing constraint is

$$\frac{\hat{L}}{R} \equiv m_t \left(q_{max} + B_{max}\right). \tag{A.24}$$

Using (A.22), (A.23), (A.24), and the definition of $\tilde{L}$, the condition $\hat{L} \leq \tilde{L}$ is satisfied if:

$$m_t \leq \overline{m} \equiv \frac{[zF_k]_{min} - \delta(R-1)}{[zF_k]_{max} + (R-1)B_{max}}, \tag{A.25}$$

for arbitrarily small δ. Given that B_{max} is finite, $\overline{m} \in (0,1)$. Condition (A.25) tells us that there is a threshold $\overline{m}$ for the collateralizable portion of the firm value, m, such that an equilibrium with a rational bubble exists. Alternatively, for certain m, there is a maximum bubble that can be injected in the economy and supported in equilibrium.

Now, suppose that $\overline{B}_t$, M_t, and, thus, B_t do not converge, which can be the case when $\mu_{t'} = 0$ for all $t' > t$. If B_{max} does not exist (i.e. it is infinite), then the borrowing constraint does not bind and $B_t = 0$ from the transversality condition (A.21). However, it follows by Aiyagari (1994) that in this (bubbleless) economy the borrowing constraint will bind infinitely often. Hence, we can perturb the economy by adding a small bubble (and sufficiently smaller than B_{max}) and support a rational bubble equilibrium where $\mu_{t'} > 0$ infinitely often; a contradiction. In other words, the borrowing constraint will always bind infinitely often, and a stochastic bubble can be supported in equilibrium.

A.5 Steps for equilibrium selection

The equilibrium selection procedure consists of the following three steps:

1. Given current states (L_t, z_t, m_t, b_t) compute all conceivable bubble values at $t+1$ denoted by $\tilde{b}_{t+1}$, conditional on the bubble not bursting, for all combinations of future states $(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1})$ using $\tilde{b}_{t+1} = b_t U_{c,t}/(\beta\pi(1 + m_t\mu_t)U_{c,t+1})$ for the growth of the bubble; c_t and μ_t can be computed from $c(L_t, z_t, m_t, b_t)$ and $\mu(L_t, z_t, m_t, b_t)$, while c_{t+1} can take many different values, $c(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1})$, given all the possible combinations of future states $(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1})$. Thus, we obtain all conceivable values for the future bubbles, $\tilde{b}_{t+1}$, as a function of the current states (L_t, z_t, m_t, b_t) and all possible future states $(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1})$. We denote the function yielding these values as $\tilde{b}_{t+1} \equiv \tilde{b}(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1}, L_t, z_t, m_t, b_t)$.

2. $\tilde{b}_{t+1}$ gives every conceivable bubble value at $t+1$ that agents may expect conditional on the bubble not bursting. Consistency of expectations then requires that $\tilde{b}(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t) = b^*_{t+1}$, where b^*_{t+1} is the state for the bubble that yields the same value at which the bubble will endogenously grow. That is, we have to solve a fixed point problem to obtain $b^*_{t+1} \equiv b^*(L_{t+1}, z_{t+1}, m_{t+1}, L_t, z_t, m_t, b_t)$ that determines the value of the bubble state at $t+1$ given $L_{t+1}, L_t, z_t, m_t, b_t$, which are known at t, along with realizations for the exogenous variables z_{t+1} and m_{t+1}. It is easy to see that such a b^*_{t+1} exists. Consider a range of possible value $[0, b_{max} + \epsilon]$ for b_{t+1}; b_{max} is the maximum value that the bubble can grow, which is finite (see Proposition 2) and $\epsilon > 0$ is small. $\tilde{b}$ is continuous and is larger than zero as $b_{t+1} \to 0$, because consumption is finite (i.e. $U_{c,t}$ is not zero) and bounded away from zero (i.e. $U_{c,t+1}$ is not infinite). Moreover, $\tilde{b}$ is smaller that b_{t+1} as $b_{t+1} \to b_{max} + \epsilon$, because it is bounded from above by b_{max}. So a solution $b^*_{t+1} \in (0, b_{max} + \epsilon)$ exists. As long as $\tilde{b}$ is not S-shaped, the solution is unambiguously

[33]Expanding the expectation E_t to only consider the states that the bubble does not burst does not alter the analysis as, in that case, we can define $\overline{B}_t = \pi^t \beta^t U_{c,t} B_t$, which is a super-martingale. Hence, given that $\pi < 1$, B_t can remain finite and continue to vary randomly as explained above.

unique. This is the case in our quantitative analysis.[34] Naturally, if the bubble bursts at $t+1$, we set $b^*_{t+1} = 0$.

3. By replacing b_{t+1} with b^*_{t+1} in the policy functions for period-t+1 variables, the expectation terms in (18) and (19) can be expressed by introducing a new set of policy functions $\mathcal{H}_u$, $\mathcal{H}_q$ that depend only on current state variables $\{L_t, z_t, m_t, b_t\}$ and future state variables $\{L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}\}$. In particular, we set

$$E_t\left(U_{c,t+1}\right) \equiv E_t\mathcal{H}_u(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t) \tag{A.26}$$

and

$$E_t\{U_{c,t+1}[z_{t+1}F_{k,t+1} + q_{t+1}(1 + m_{t+1}\mu_{t+1})]\} \equiv E_t\mathcal{H}_q(L_{t+1}, z_{t+1}, m_{t+1}, b^*_{t+1}, L_t, z_t, m_t, b_t). \tag{A.27}$$

A.6 Optimality conditions for planner's problem

The social planner's optimality conditions are the following:

$$\text{wrt } c_t: \quad \lambda^p_t = U_{c,t} - \xi_t q_t U_{cc,t} + \xi_t\beta E_t \frac{d\mathcal{H}_{q,t+1}}{db^*_{t+1}}\frac{db^*_{t+1}}{dc_t}, \tag{A.28}$$

$$\text{wrt } q_t: \quad \xi_t U_{c,t} = m_t\mu^p_t, \tag{A.29}$$

$$\text{wrt } L_{t+1}: \quad \lambda^p_t = \beta R E_t\left(\lambda^p_{t+1} + \xi_t\frac{d\mathcal{H}_{q,t+1}}{dL_{t+1}}\right) + \mu^p_t, \tag{A.30}$$

$$\begin{aligned}\text{wrt } v_t: \quad & \lambda^p_t(z_tF_{v,t} - p^v) - \theta p^v\mu^p_t - \mu^p_t m_t\frac{b_t}{(1+m_t\mu(l_t,v_t))^2}\frac{z_tF_{vv.t}}{\theta p^v} \\ & + \xi_t\beta E_t\frac{d\mathcal{H}_{q,t+1}}{db^*_{t+1}}\frac{db^*_{t+1}}{dv_t} - \zeta_t z_t F_{vl,t} = 0,\end{aligned} \tag{A.31}$$

$$\begin{aligned}\text{wrt } l_t: \quad & -U_{c,t}G_{l,t} + \lambda^p_t z_t F_{l,t} + \xi_t U_{cc,t}G_{l,t}q_t - \mu^p_t m_t\frac{b_t}{(1+m_t\mu(l_t,v_t))^2}\frac{z_tF_{vl.t}}{\theta p^v} \\ & + \xi_t\beta E_t\frac{d\mathcal{H}_{q,t+1}}{db^*_{t+1}}\frac{db^*_{t+1}}{dl_t} - \zeta_t(z_tF_{ll,t} - G_{ll,t}) = 0,\end{aligned} \tag{A.32}$$

where $d\mathcal{H}_{q,t+1}/db^*_{t+1}\cdot(db^*_{t+1}/d(\cdot))$ and $d\mathcal{H}_{q,t+1}/dL_{t+1}$ capture how period-t choices of the planner affect the period-t+1 endogenous state variables and through them the forward looking terms that matter for the determination of q_t. Hence, the planner internalizes how her choices affect the actions of future planners, reflecting the time-consistent nature of the policy rule.

By totally differentiating the bubble accumulation condition (25), we obtain

$$\frac{db^*_{t+1}}{dc_t} = \frac{b^*_{t+1}}{1 + b^*_{t+1}\frac{U^*_{cc,t+1}}{U^*_{c,t+1}}\frac{dc^*_{t+1}}{db^*_{t+1}}}\frac{U_{cc,t}}{U_{c,t}}, \tag{A.33}$$

[34] The first and second derivatives of $\tilde{b}$ with respect to b_{t+1} are $d\tilde{b}/db_{t+1} = -\tilde{b}\cdot U_{cc,t+1}/U_{c,t+1}\cdot dc_{t+1}/db_{t+1}$ and $d^2\tilde{b}/db^2_{t+1} = -\tilde{b}/U_{c,t+1}\left[U_{ccc,t+1}\cdot(dc_{t+1}/db_{t+1})^2 + U_{cc,t+1}\cdot d^2c_{t+1}/db^2_{t+1}\right]$, respectively. If $dc_{t+1}/db_{t+1} < 0$, then $d\tilde{b}/db_{t+1} < 0$ and the solution b^*_{t+1} is unique irrespective of the second derivative of $\tilde{b}$. Otherwise, $\tilde{b}$ is increasing but it is not S-shaped if $d^2\tilde{b}/db^2_{t+1}$ does not change sign, for which it suffices that $d^2c_{t+1}/db^2_{t+1} < 0$ given that $U_{ccc,t+1} > 0$. In other words, consumption can increase in the size of the bubbles but at a decreasing rate. If none of these sufficient conditions to exclude multiple b^*_{t+1} hold, then one could use a refinement according to which agents coordinate on the highest b^*_{t+1} solution and still be able to follow our steps to compute equilibrium.

$$\frac{db_{t+1}^*}{dv_t} = -\frac{b_{t+1}^*}{1+b_{t+1}^*\frac{U_{cc,t+1}^*}{U_{c,t+1}^*}\frac{dc_{t+1}^*}{db_{t+1}^*}}\frac{z_tF_{vv,t}}{\theta p^v}\frac{m_t}{1+m_t\mu(l_t,v_t)}, \tag{A.34}$$

and

$$\frac{db_{t+1}^*}{dl_t} = -\frac{db_{t+1}^*}{dc_t}G_{l,t} - \frac{b_{t+1}^*}{1+b_{t+1}^*\frac{U_{cc,t+1}^*}{U_{c,t+1}^*}\frac{dc_{t+1}^*}{db_{t+1}^*}}\frac{z_tF_{vl,t}}{\theta p^v}\frac{m_t}{1+m_t\mu(l_t,v_t)}, \tag{A.35}$$

where c_{t+1}^* is given by policy function $c(L_{t+1}, z_{t+1}, m_{t+1}, b_{t+1}^*)$. We proceed with solving for the Lagrange multipliers in the planner's first-order conditions.

Using (A.28), (A.29), (A.31), (A.32), (A.33), (A.34), (A.35), $z_tF_{l,t} = G_{l,t}$, $\mu(l_t, v_t) = (z_tF_{v,t} - p^v)/(\theta p^v)$, $d\mu(l_t, v_t)/dl_t = z_tF_{vl,t}/(\theta p^v)$, and $d\mu(l_t, v_t)/dv_t = z_tF_{vv,t}/(\theta p^v)$, we get that

$$\mu_t^p = \lambda_t^p\mu(l_t, v_t) + \zeta_t\mathcal{Z}_t \tag{A.36}$$

and

$$\zeta_t = -\frac{\mu_t^p}{z_tF_{ll,t}-G_{ll,t}}\frac{z_tF_{vl,t}}{\theta p^v}\frac{m_t}{1+m_t\mu(l_t,v_t)}\left(\frac{m_t}{U_{c,t}}\mathcal{X}_t + \frac{b_t}{1+m_t\mu(l_t,v_t)}\right), \tag{A.37}$$

where

$$\mathcal{Z}_t = \frac{z_tF_{ll,t}-G_{ll,t}}{\theta p^v}\frac{F_{vv,t}}{F_{vl,t}} - \frac{z_tF_{vl,t}}{\theta p^v} \tag{A.38}$$

and

$$\mathcal{X}_t = \beta E_t\frac{d\mathcal{H}_{q,t+1}}{db_{t+1}^*}\frac{b_{t+1}^*}{1+b_{t+1}^*\frac{U_{cc,t+1}^*}{U_{c,t+1}^*}\frac{dc_{t+1}^*}{db_{t+1}^*}}. \tag{A.39}$$

We can now derive an expression for λ_t^p as follows. First, substitute (A.29) in (A.28) to eliminate ξ_t. Then substitute (A.36) in the second term in (A.28), and use (A.37) to eliminate ζ_t. Note also that

$$dB_t/dl_t = -(z_tF_{vl,t}m_tb_t)/(\theta p^v(1+m_t\mu(l_t,v_t))^2).$$

Finally, using (A.33) and (A.39) in (A.29), yields

$$\lambda_t^p = U_{c,t} + \mathcal{F}_tq_t + \mathcal{W}_t\frac{dB_t}{dl_t} + \mathcal{V}_t\beta E_t\frac{d\mathcal{H}_{q,t+1}}{db_{t+1}^*}\frac{db_{t+1}^*}{dc_t} \tag{A.40}$$

where

$$\mathcal{F}_t = -\frac{m_t\mu(l_t,v_t)U_{cc,t}}{1+m_t\mu(l_t,v_t)\frac{U_{cc,t}}{U_{c,t}}q_t}, \tag{A.41}$$

$$\mathcal{W}_t = -\frac{m_t\mu_t^p\mathcal{Z}_tq_t}{z_tF_{ll,t}-G_{ll,t}}\frac{U_{cc,t}}{U_{c,t}}\frac{1}{1+m_t\mu(l_t,v_t)\frac{U_{cc,t}}{U_{c,t}}q_t}, \tag{A.42}$$

and

$$\mathcal{V}_t = \left(\frac{m_t}{1+m_t\mu(l_t,v_t)}\frac{z_tF_{vl,t}}{\theta p^v}\frac{m_tq_t\mathcal{Z}_t}{z_tF_{ll,t}-G_{ll,t}}+1\right)\frac{m_t\mu_t^p}{U_{c,t}}\frac{1}{1+m_t\mu(l_t,v_t)\frac{U_{cc,t}}{U_{c,t}}q_t}. \tag{A.43}$$

(A.40) is equation (34) reported in the main body of the paper. In some occasions it is also useful to note that $\mathcal{W}_t$ can be written as $\mathcal{W}_t = F_tq_t\mathcal{W}_t'$ with

$$\mathcal{W}_t' = \frac{\mu_t^p}{\mu(l_t,v_t)U_{c,t}}\frac{\mathcal{Z}_t}{z_tF_{ll,t}-G_{ll,t}} < 0. \tag{A.44}$$

First, note that $\mathcal{F}_t > 0$, since $U_{cc,t} < 0$, as long as $1 + m_t\mu(l_t, v_t)U_{cc,t}/U_{c,t}q_t > 0$, which is necessary to obtain a positive shadow value of income, λ_t^p, for both the bubbleless and bubbly economies; this is also the case in Bianchi-Mendoza and we verify it in our quantitative solution.

Next, from the properties of the Cobb-Douglas production function, $F(k_t, v_t, l_t) = k_t^{a_k} v_t^{a_v} l_t^{a_l}$, we have that $F_{vv,t} = (\alpha_v - 1)\alpha_v k_t^{\alpha_k} v_t^{\alpha_v - 2} l_t^{\alpha_l}$, $F_{vl,t} = \alpha_v \alpha_l k_t^{\alpha_k} v_t^{\alpha_v - 1} l_t^{\alpha_l - 1} = \alpha_v F_{l,t}/v_t$, $F_{ll,t} = (\alpha_l - 1)\alpha_l k_t^{\alpha_k} v_t^{\alpha_v} l_t^{\alpha_l - 2}$, where a_v and a_l are the shares of the intermediate good and labor in production with $1 - a_v - a_l > 0$. Using these, we can re-write $\mathcal{Z}_t$ as

$$\mathcal{Z}_t = \frac{(1 - a_l - a_v) z_t F_{l,t} + (1 - a_v) G_{ll,t} l_t}{a_l \theta p^v v_t},$$

i.e. $\mathcal{Z}_t$ is positive. This means that $\mathcal{W}_t$ in (A.42) is negative because $z_t F_{ll,t} - G_{ll,t} < 0$ and $U_{cc,t} < 0$, Thus, also the third term in (A.40), $\mathcal{W}_t dB_t/dl_t$, is positive overall because $dB_t/dl_t < 0$. However, we cannot unambiguously sign $\mathcal{V}_t$ in (A.43) and, hence, the fourth term in (A.40).

$\mathcal{F}_t$, $\mathcal{W}_t$, and $\mathcal{V}_t$ depend on λ_t^p and μ_t^p. In order to express them just in terms of the other equilibrium variables we use (A.28), (A.29), (A.36), and (A.37) to obtain λ_t^p and μ_t^p as the solution to the following system of equations:

$$\lambda_t^p = U_{c,t} - \frac{m_t \mu_t^p}{U_{c,t}} q_t U_{cc,t} + \frac{m_t \mu_t^p}{U_{c,t}} \frac{U_{cc,t}}{U_{c,t}} \mathcal{X}_t \tag{A.45}$$

$$\mu_t^p = \frac{\lambda_t^p \mu(l_t, v_t)}{1 + \frac{m_t}{1 + m_t \mu(l_t, v_t)} \frac{z_t F_{vl,t}}{\theta p^v} \left(\frac{m_t \mathcal{X}_t}{U_{c,t}} + B_t \right) \frac{\mathcal{Z}_t}{z_t F_{ll,t} - G_{ll,t}}} \tag{A.46}$$

Finally, note that the third and fourth terms in (A.40) associated with the bubble are zero if $b_t = 0$, while all the last three terms associated with externalities are zero if the collateral constraint does not bind, as $\mu_t = \mu_t^p = \mathcal{W}_t = \mathcal{V}_t = 0$ irrespective of the presence of the bubble.

A.7 Numerical Algorithm

Competitive equilibrium. We solve for the CE using an Euler-equation iteration algorithm. In each iteration, we solve the system of equations presented below in a recursive form for: 150 states for the level of debt denoted by L; 6 exogenous states denoted by ω (3 states for productivity, z, and 2 states for the pledgeable fraction of collateral, m); and and 10 states for stock market bubble states denoted by b. We use the convention that L', ω', and b' denote the states one period ahead. Formally, we solve for the policy functions $\{\tilde{L}(L,\omega,b), c(L,\omega,b), q(L,\omega,b), l(L,\omega,b), v(L,\omega,b), \mu(L,\omega,b), B(L,\omega,b), H_U(L,\omega,b), H_q(L,\omega,b)$, such that the equilibrium conditions below are satisfied

$$c(L,\omega,b) + L + p^v v(L,\omega,b) = zF(1, v(L,\omega,b), l(L,\omega,b)) + \frac{\tilde{L}(L,\omega,b)}{R}, \tag{A.47}$$

$$\frac{\tilde{L}(L,\omega,b)}{R} + \theta p^v v(L,\omega,b) \le m(q(L,\omega,b) + B(L,\omega,b)), \tag{A.48}$$

$$b = (1 + m\mu(L,\omega,b)) B(L,\omega,b), \tag{A.49}$$

$$(1 - \mu(L,\omega,b)) U_c(c(L,\omega,b) - G(l(L,\omega,b))) = \beta R E_{\omega'|\omega} H_U(L', \omega', b^*, L, \omega, b), \tag{A.50}$$

$$q(L,\omega,b) U_c(c(L,\omega,b) - G(l(L,\omega,b))) = \beta E_{\omega'|\omega} H_q(L', \omega', b^*, L, \omega, b), \tag{A.51}$$

$$zF_l(1, v(L,\omega,b), l(L,\omega,b)) = G_l(L,\omega,b), \tag{A.52}$$

$$zF_v(1, v(L,\omega,b), l(L,\omega,b)) = p^v(1 + \theta\mu(L,\omega,b)), \tag{A.53}$$

$$H_U(L',\omega',b^*,L,\omega,b) = U_c(c(L',\omega',b^*) - G(l(L',\omega',b^*))), \tag{A.54}$$

$$\begin{aligned} H_q(L',\omega',b^*,L,\omega,b) = U_c(c(L',\omega',b^*) - G(l(L',\omega',b^*)))\cdot \\ (z'F_k(1, v(L',\omega',b^*), l(L',\omega',b^*)) + (1 + m'\mu(L',\omega',b^*))q(L',\omega',b^*)), \end{aligned} \tag{A.55}$$

$\tilde{L}(L,\omega,b)$ is the new borrowing and $b^* \equiv b^*(L',\omega',L,\omega,b)$ is the solution to $\bar{b}(L',\omega',b^*,L,\omega,b) = b^*$, where $\bar{b}(L',\omega',b',L,\omega,b)$ is a function that generates all possible values for the bubble in the future given current states L,ω,b and all possible future states L',ω,b'. This function is derived by the bubble accumulation process $\bar{b}(L',\omega',b',L,\omega,b) = B(L,\omega,b)U_c(c(L,\omega,b) - G(l(L,\omega,b)))/(\beta\pi U_c(c(L',\omega',b') - G(l(L',\omega',b')))$.

The algorithm proceeds in the following steps:

1. For each state (L,ω,b), conjecture policy functions $L' = \tilde{L}(L,\omega,b)$, $c(L,\omega,b)$, $q(L,\omega,b)$, $l(L,\omega,b)$, $v(L,\omega,b)$, $\mu(L,\omega,b)$, $B(L,\omega,b)$.
2. Given conjectures step 1, compute $\bar{b}(L',\omega',b',L,\omega,b)$ for all combinations (L',ω',b') at each (L,ω,b).
3. For each (L',ω',L,ω,b), set $b^*(L',\omega',L,\omega,b) = argmin_{b'}|\bar{b}(L',\omega',b',L,\omega,b) - b'|$, i.e. the bubble state that is closest to the value that the bubble should go to.
4. For (L',ω',L,ω,b), generate conjectures for H_U, and H_q using the value of $b^*(L',\omega',L,\omega,b)$ in step 3.
5. For (L,ω,b), use the conjectures in step 1 and the conjectures for H_U, and F_q in step 4 to obtain new conjectures for (current) policy functions $\tilde{L}(L,\omega,b)$, $c(L,\omega,b)$, $q(L,\omega,b)$, $l(L,\omega,b)$, $v(L,\omega,b)$, $\mu(L,\omega,b)$, $B(L,\omega,b)$. We distinguish between cases that the borrowing constraint binds and does not bind in the present:
   i. First, assume that the borrowing constraint (A.48) binds and solve for the current policy functions. Then, check that $\mu(L,\omega,b) > 0$ using equation (A.50). If this is true, proceed to step 6; otherwise move to substep ii.
   ii. If the borrowing constraint in the present does not bind, solve the system of equations above for the current policy functions by setting $\mu(L,\omega,b) = 0$.
6. Use the optimal policy functions from substeps 5i or 5ii to update the (conjectured) policy functions in step 1.
7. Stop when convergence is achieved, i.e. when for two consecutive iterations $i-1$ and i it holds that $\sup_{L,\omega} ||x_i(L,\omega) - x_{i-1}(L,\omega)|| < \varepsilon$, where $x = \tilde{L}, c, q, l, v$. We set $\varepsilon = 10^{-4}$.

Social planner. We solve for the SP policy functions using a value function iteration, nested fixed point algorithm. In each iteration we solve for the value function using a fixed-grid optimization procedure as an inner loop. In the outer loop, we update future policies given the solution to the Bellman equation from the inner loop. As in Klein, Krusell and Ríos-Rull (2008) and Bianchi and Mendoza (2018), this procedure delivers time-consistent policies. The detailed steps are described below.

The value function representation of the SP's optimization problem is:

$$V(L,\omega,b) = \max_{\tilde{L},c,l,v,q,B,\mu} \big(U(c(L,\omega,b) - G(l(L,\omega,b))) + \beta E_{\omega'|\omega}[H_V(L',\omega',b^*,L,\omega,b)]\big) \quad \text{(A.56)}$$

subject to (A.47)-(A.49) and (A.51)-(A.55); that is, all the optimality conditions in the competitive equilibrium constitute constraints in the planner's problem apart for condition (A.50) with respect to $\tilde{L}(L,\omega,b)$, which is missing the externality terms.

$H_V(L',\omega',b^*,L,\omega,b)$ is a policy function that yields the future value function and is computed similarly to functions H_U, and H_q, i.e.

$$H_V(L',\omega',b^*,L,\omega,b) = V(L',\omega',b^*). \quad \text{(A.57)}$$

The algorithm proceeds in the following steps:

1. In the outer loop:

   i. Define policies $V(L,\omega,b)$, $\tilde{L}(L,\omega,b)$, $c(L,\omega,b)$, $q(L,\omega,b)$, $l(L,\omega,b)$, $v(L,\omega,b)$, $\mu(L,,\omega,b)$, $B(L,\omega,b)$ as the updated solution from the previous iteration (see step 3 below) or the policy functions from the CE solution for the first iteration.

   ii. Compute $\bar{b}(L',\omega',b',L,\omega,b)$ for all combinations (L',ω',b') at each (L,ω,b).

   iii. For each (L',ω',L,ω,b), set $b^*(L',\omega',L,\omega,b) = argmin_{b'}|\bar{b}(L',\omega',b',L,\omega,b) - b'|$, i.e. the bubble state that is closest to the value that the bubble should go to.

   iv. For (L',ω',L,ω,b), generate conjectures for H_U, H_q, and H_V using the value of $b^*(L',\omega',L,\omega,b)$ in step 1iii.

2. In the inner loop, for each (L,ω,b), use the conjectures in step 1 to obtain new conjectures for (current) policy functions $V(L,\omega,b)$, $\tilde{L}(L,\omega,b)$, $c(L,\omega,b)$, $q(L,\omega,b)$, $l(L,\omega,b)$, $v(L,\omega,b)$, $\mu(L,\omega,b)$, $B(L,\omega,b)$ that satisfy (A.47)-(A.49) and (A.51)-(A.55). We distinguish between the case that the borrowing constraint binds and the case that the borrowing constraint does not bind today:

   i. First, assume that the borrowing constraint (A.48) does not bind, i.e. $\mu(L,\omega) = 0$. Then, the objective is to find the level of $\tilde{L}(L,\omega,b)$ that maximizes (A.56). To do that, we first solve for v and l using (A.52) and (A.53) and substitute out consumption using the budget constraint (A.47). Then, we compute $\tilde{L}(L,\omega,b)$ by calculating (A.56) for a subgrid of 5000 values of $\tilde{L}$ and choosing the value with the highest $V(L,\omega,b)$: $\tilde{L}$ matters for $V(L,\omega,b)$ not only because it determines current utility $U(c(L,\omega,b) - G(l(L,\omega,b)))$, but also because it is the future state variable, i.e. $L' = \tilde{L}(L,\omega,b)$. Thus, its choice determines the level of the continuation value $H_V(L',\omega',b^*,L,\omega,b)$. The policy function $H_V(L',\omega',b^*,L,\omega,b)$ assigning a value for different values L' is taken as given from the outer loop in step 1, but, in the inner loop, we choose the value of L' ($\tilde{L}$) that maximizes the sum of current utility and the continuation value. Finally, check if the optimal choice of $\tilde{L}(L,\omega,b)$ is lower than the borrowing limit computed from (A.48). If this is true, proceed to step 3; otherwise move to substep ii.

   ii. Solve for the current policy functions given future polices from step 1, but set (A.48) to hold with equality. For each point on the subgrid of $\tilde{L}$ values, calculate corresponding values of c, q, l, v, μ, B satisfying equations (A.47)-(A.49) and (A.51)-(A.55). Finally, choose the level of $\tilde{L}$ for which $V(L,\omega,b)$ is the highest similar to substep i above.

3. Use the optimal policy functions from substeps 2i or 2ii to update the conjectured policy functions in step 1.

4. Stop when convergence is achieved, i.e. when for two consecutive iterations $i-1$ and i it holds that $\sup_{L,\omega} ||x_i(L,\omega) - x_{i-1}(L,\omega)|| < \varepsilon$, where $x = \tilde{L}, c, q, l, v$, and $\varepsilon = 10^{-4}$.